Mathematical Methods and Algorithms For Signal Processing: Solutions Manual

Version 1.0

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Preface

It is hoped that the solutions for *Mathematical Methods and Algorithms for Signal Processing* turns out to be helpful to both instructor and student.

In the solutions, an attempt has been made to display key concepts leading to the solution, without necessarily showing all of the steps. Depending on the problem (and my energy available for typing), varying degrees of detail are shown. In some cases, a very complete solution is provided; in others, simply some hopefully helpful hints. Wherever I found the computer to be of help in obtaining the solution, I have attempted to provide the input to the computer, either in MATLAB or MATHEMATICA; this provides, I think, more useful information than simply giving a numeric or symbolic answer.

While the vast majority of the problems in the book have solutions presented here there remain a few which do not. These include some computer problems (although many of the computer exercises *do* have solutions), some which involve simulations (in which case a starting point is frequently suggested), and some which would be typographically difficult.

While I have attempted to be accurate in my solutions, the very size of the document and the amount of detail leaves me (in)secure in the knowledge that there must still be errors there. I have been greatly assisted in tracking down errors as I have taught through the text in our Fall Semester 6660 class (Mathematical Methods for Signal Processing), but we haven't made it through the book!

I gratefully acknowledge the assistance of Wynn Stirling with some problems from chapter 11 and chapter 13.

Feedback is important to me. If errors or improved solutions are found, please let me know at

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Sooner or later they will be incorporated into these solutions. And any new problems that are found that fit within a chapter are also of interest.

Contents

List of Algorithms

Chapter 1

Introduction

1.4-1 (a) $z_3 = z_1 z_2 = e + jf = (ac - bd) + j(ad + bc)$.

$$\begin{bmatrix} c & -d \\ d & c \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ac - bd \\ ad + bc \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

(b)

$$e = (a - b)d + a(c - d) = ac - bd$$
$$f = (a - b)d + b(c + d) = ad + bc$$

(c)

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} (c-d) & 0 & 0 \\ 0 & (c+d) & 0 \\ 0 & 0 & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} (c-d) & 0 & 0 \\ 0 & (c+d) & 0 \\ 0 & 0 & d \end{bmatrix} \begin{bmatrix} a \\ b \\ a-b \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a(c-d) \\ b(c+d) \\ d(a-b) \end{bmatrix}$$
$$= \begin{bmatrix} a(c-d) + d(a-b) \\ b(c+d) + d(a-b) \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}.$$

1.4-2

$$H(z)(1 - pz^{-1})^r = k_0 + k_1(1 - pz{-}1) + k_2(1 - pz^{-1})^2 + k_3(1 - pz^{-3})^3 + \cdots + k_{r-1}(1 - pz^{-1})^{r-1}$$

so that

$$H(z)(1 - pz^{-1})^r\big|_{z=p} = k_0$$

Also

$$\frac{d}{dz^{-1}} H(z)(1 - pz^{-1})^r = -pk_1 - 2pk_2(1 - pz^{-1}) - 3pk_3(1 - pz^{-3})^2 - \cdots - (r-1)pk_{r-1}(1 - pz^{-1})^{r-2}$$

so that

$$\left.\frac{d}{dz^{-1}} H(z)(1 - pz^{-1})^r\right|_{z=p} = -pk_1$$

from which

$$k_1 = -1/p \left.\frac{d}{dz^{-1}} H(z)(1 - pz^{-1})^r\right|_{z=p}.$$

Continuing similarly, (1.8) follows.

1.4-3 (a)

$$H(z) = \frac{1 - 3z^{-1}}{1 - 1.5z^{-1} + .56z^{-2}} = \frac{1 - 3z^{-1}}{(1 - .8z^{-1})(1 - .7z^{-1})} = \frac{k_1}{(1 - .8z^{-1})} + \frac{k_2}{(1 - .7z^{-1})}$$

where, by cover-up and plug-in (CUPI)

$$k_1 = \left.\frac{(1 - 3z^{-1})}{(1 - .7z^{-1})}\right|_{z=.8} = -22 \qquad k_2 = \left.\frac{(1 - 3z^{-1})}{(1 - .8z^{-1})}\right|_{z=.7} = 23.$$

MATLAB results (with some reformatting)::

```
[r,p] = residuez([1 -3],[1 -1.5 .56])
r =[-22.0000  23.0000]
p =[0.8000 0.7000]
```

(b)

$$H(z) = \frac{1-5z^{-1}-6z^{-2}}{1-1.5z^{-1}+.56z^{-2}} = \frac{1-5z^{-1}-6z^{-2}}{(1-.8z^{-1})(1-.7z^{-1})}$$
$$= -10.7143 + \frac{11.7143-21.-714z^2}{(1-.8z^{-1})(1-.7z^{-1})}$$
$$= -10.7143 + \frac{k_1}{(1-.8z^{-1})} + \frac{k_2}{(1-.7z^{-1})}$$

where, by CUPI,

$$k_1 = \left.\frac{1-5z^{-1}-6z^{-2}}{(1-.7z^{-1})}\right|_{z=.8} = -117 \qquad k_2 = \left.\frac{1-5z^{-1}-6z^{-2}}{(1-.8z^{-1})}\right|_{z=.z} = 128.7143$$

MATLAB results:

```
[r,p,k] = residuez([1 -5 -6],[1 -1.5 .56])
r =[-117.0000  128.7143]
p =[0.8000 0.7000]
k = -10.7143
```

(c)

$$H(z) = \frac{2-3z^{-1}}{(1-.3z^{-1})^2} = \frac{k_0}{(1-.3z^{-1})^2} + \frac{k_1}{(1-.3z^{-1})}$$

where

$$k_0 = 2-3z^{-1}\big|_{z=.3} = -8 \qquad k_1 = \left.\frac{-1}{.3}\frac{d}{dz^{-1}}2-3z^{-1}\right|_{z=0.3} = 10.$$

MATLAB results:

```
[r,p] = residuez([2 -3],conv([1 -.3],[1,-.3]))
r =[ 10.0000 -8.0000]
p =[0.3000  0.3000]
```

(d)

$$H(z) = \frac{5-6z^{-1}}{(1-.3z^{-1})^2(1-.4z^{-1})} = \frac{k_0}{(1-.3z^{-1})^2} + \frac{k_1}{(1-.3z^{-1})} + \frac{k_2}{(1-.4z^{-1})}$$

where, by CUPI,

$$k_0 = \left.\frac{5-6z^{-1}}{(1-.4z^{-1})}\right|_{z=.3} = 45 \qquad k_2 = \left.\frac{5-6z^{-1}}{(1-.3z^{-1})^2}\right|_{z=.4} = -160$$

and

$$k_1 = -\frac{1}{.3}\frac{d}{dz^{-1}}\left.\frac{5-6z^{-1}}{(1-.4z^{-1})}\right|_{z=.3} = 120$$

MATLAB results:

```
[r,p] = residuez([5 -6],conv(conv([1 -.3],[1,-.3]),[1 -.4]))
r =[ -160 120 45]
p =[0.4000  0.3000 0.3000]
```

1.4-4 (a) $X(z) = \sum_t x[t]z^{-t}$, so that

$$\frac{d}{dz}X(z) = \sum_t (-t)x[t]z^{-t-1},$$

so

$$-z\frac{d}{dz}X(z) = \sum_t (tx[t])z^{-t} = \mathcal{Z}[tx[t]]$$

(b) Use the previous result:

$$tp^t u[t] \leftrightarrow -z\frac{d}{dz}\frac{1}{(1-pz^{-1})}.$$

(c) $t^2p^tu[t] \leftrightarrow -z\frac{d}{dz}\frac{pz^{-1}}{(1-pz^{-1})^2}$. Using Mathematica, we readily obtain using

```
Together[-z D[p z^(-1)/(1-p z^(-1))^2,z]]
```

the answer

$$t^2p^tu[t] \leftrightarrow \frac{pz^{-1}(1+pz^{-1})}{(1-pz^{-1})^3}.$$

(d) The pole of a mode of the form $t^kp^tu[t]$ is of order $(k+1)$.

1.4-5 We have

$$r_{yy}[k] = E[y[t]\overline{y}[t-k]]$$

so that

$$\overline{r}_{yy}[-k] = E[\overline{y}[t]y[t+k]] = E[y[u]\overline{y}[u-k]] = r_{yy}[k]$$

where $u = t+k$

1.4-6 Letting $b_0 = 1$, we have

$$\begin{aligned} r_{yy}[k] &= E[\sum_{i=0}^{q}\overline{b}_i f[t-i]\sum_{j=0}^{q} b_j\overline{f}[t-j-k]] \\ &= \sum_{i=0}^{q}\sum_{j=0}^{q}\overline{b}_i b_j\sigma_f^2\delta_{i,j+k} = \sigma_f^2\sum_i \overline{b}_i b_{i-k}. \end{aligned}$$

1.4-7

$$\begin{aligned} r_{yy}[0] &= \sigma_f^2[(1)^2 + (2)^2 + 3^2] = 1.4 \\ r_{yy}[1] &= \sigma_f^2[(1)(2) + (2)(3)] = .8 \\ r_{yy}[2] &= \sigma_f^2[(1)(3)] = 0.3 \end{aligned}$$

so

$$R = \begin{bmatrix} 1.4 & .8 & .3 \\ .8 & 1.4 & .3 \\ .3 & .8 & 1.4 \end{bmatrix}.$$

1.4-8 Noting that $E[f[t+1]y[t]] = 0$, since future values of noise are uncorrelated with present values of the output, we have

$$\sigma_y^2 = E[y[t+1]y[t+1]] = E[(f[t+1] - a_1y[t])(f[t+1] - a_1y[t])] = \sigma_f^2 + a_1^2\sigma_y^2,$$

from which $\sigma_y^2(1-a_1)^2 = \sigma_f^2$, and the result follows.

1.4-9 Starting from the derivation of the Yule-Walker equations,

$$E\left[\sum_{k=0}^{p}\overline{a}_k y[t-k]\overline{y}[t-l]\right] = E[f[t]\overline{y}[t-l]]$$

set $l = 0$, obtain

$$E\left[\sum_{k=0}^{p}\overline{a}_k y[t-k]\overline{y}[t]\right] = E[f[t]\overline{y}[t]] = \sigma_f^2.$$

The result follows by conjugating both sides of the equation.

1.4-10 (Second-order AR)

(a) From the Yule-Walker equations,

$$\begin{bmatrix} r_{yy}[0] & r_{yy}[1] \\ r_{yy}[1] & r_{yy}[0] \end{bmatrix}\begin{bmatrix} -a_1 \\ -a_2 \end{bmatrix} = \begin{bmatrix} r_{yy}[1] \\ r_{yy}[2] \end{bmatrix}$$

so that

$$\begin{aligned} \begin{bmatrix} -a_1 \\ -a_2 \end{bmatrix} &= \begin{bmatrix} r_{yy}[0] & r_{yy}[1] \\ r_{yy}[1] & r_{yy}[0] \end{bmatrix}^{-1}\begin{bmatrix} r_{yy}[1] \\ r_{yy}[2] \end{bmatrix} = \frac{1}{r_{yy}[0]^2 - r_{yy}[1]^2}\begin{bmatrix} r_{yy}[0] & -r_{yy}[1] \\ -r_{yy}[1] & r_{yy}[0] \end{bmatrix}\begin{bmatrix} r_{yy}[1] \\ r_{yy}[2] \end{bmatrix} \\ &= \frac{1}{r_{yy}[0]^2 - r_{yy}[1]^2}\begin{bmatrix} r_{yy}[1](r_{yy}[0] - r_{yy}[2]) \\ r_{yy}[0]r_{yy}[2] - r_{yy}[1]^2 \end{bmatrix}. \end{aligned}$$

(b) Using $y[t+1] = f[t+1] - a_1 y[t] - a_2 y[t-1]$ we have

$$r_{yy}[1] = E[y[t]y[t+1]] = E[y[t](f[t+1] - a_1 y[t] - a_2 y[t-1])] = -a_1\sigma_y^2 - a_2 r_{yy}[1],$$

so that $(1+a_2)r_{yy}[1] = -a_1\sigma_y^2$, from which the result follows. Also,

$$r_{yy}[2] = E[y[t]y[t+2]] = E[y[t](f[t+2] - a_1 y[t+1] - a_2 y[t])] = -a_1 r_{yy}[1] - a_2\sigma_y^2.$$

Substituting from the previous result for $r_{yy}[1]$ we have

$$r_{yy}[2] = -a_1\left(\frac{-a_1}{1+a_2}\sigma_y^2\right) - a_2\sigma_y^2,$$

from which the result follows.

(c) By (1.76) we have

$$\sigma_f^2 = r_{yy}[0] + a_1 r_{yy}[1] + a_2 r_{yy}[2].$$

Substituting for $r_{yy}[1]$ and $r_{yy}[2]$, we have

$$\sigma_f^2 = \sigma_y^2(1 - \frac{a_1^2}{a+a_2} + \frac{a_1^2 a_2}{1+a_2} - a_2^2).$$

Solving for σ_y^2, the result follows.

(d) The solution of the difference equation is of the form

$$r_{yy}[k] = c_1 p_1^k + c_2 p_2^k,$$

where c_1 and c_2 are found according to the initial conditions:

$$r_{yy}[0] = c_1 + c_2$$
$$r_{yy}[1] = c_1 p_1 + c_2 p_2$$

from which we obtain

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ p_1 & p_2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -a_1/(1+a_2) \end{bmatrix} \sigma_y^2 = \frac{\sigma_y^2}{(p_2 - p_1)(1+a2)} \begin{bmatrix} p_2(1+a2) + a_1 \\ -p_1 - p_1 a_2 - a_1 \end{bmatrix}$$

1.4-11 Use the results of the previous problem. The computations are represented in the following MATLAB function:

Algorithm 1.1 AR(2) coefficients to autocorrelation

```
function [sigma2,r1,r2] = ator2(a1,a2,sigmaf2)
% function [sigma2,r1,r2] = ator2(a1,a2,sigmaf2)
% Given the coefficients from a 2nd-order AR model
% y[t+2] + a1 y[t+1] + a2 y[t] = f[t+2],
% where f has variance sigmaf2, compute sigma_y^2, r[1], and r[2].

sigma2 = sigmaf2* (1+a2)/((1-a2)*((1+a2)^2 - a1^2));
r1 = -sigma2*a1/(1+a2);
r2 = sigma2*(a1^2/(1+a2) - a2);
```

Then the answer is provided by

```
a1 = -.7; a2=.12;  sigmaf2 = .1;
[sigmay2,r1,r2] = ator2(a1,a2,sigmaf2)
```

yielding

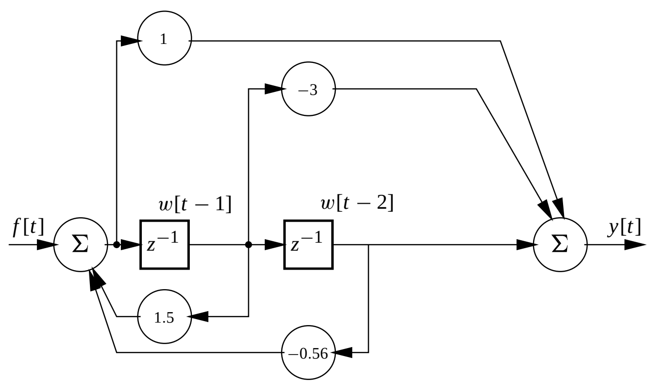
$$\sigma_y^2 = 0.1665 \qquad r_1 = 0.1041 \qquad r_2 = 0.0529$$

1.4-12 The filter output is $x[t] = \sum_m y[t-m]h[m]$. The average power is

$$E[x[t]x[t]] = E\left[\sum_m y[t-m]h[m]\sum_n y[t-n]h[n]\right] = \sum_m\sum_n h[m]E[y[t-m]y[t-n]]h[n]$$
$$= \mathbf{h}^T R\mathbf{h}.$$

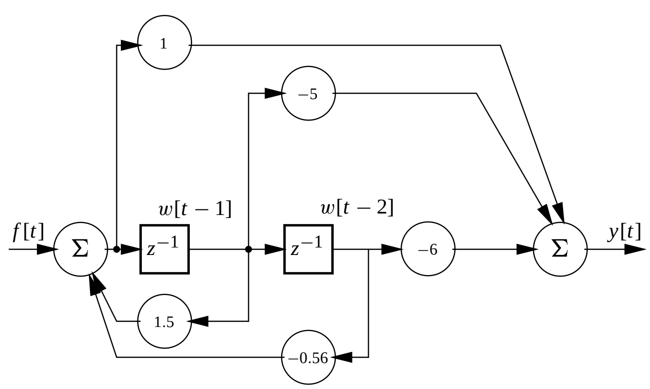
1.4-13 (a) State variable form:

$$A = \begin{bmatrix} 0 & 1 \\ -0.56 & 1.5 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} -0.56 \\ -1.5 \end{bmatrix}$$



(b) State variable form:

$$A = \begin{bmatrix} 0 & 1 \\ -0.56 & 1.5 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} -6.5 \\ -3.5 \end{bmatrix}$$



1.4-14 Observer canonical form:

(a) Assume for notational ease that $p = q$. We have

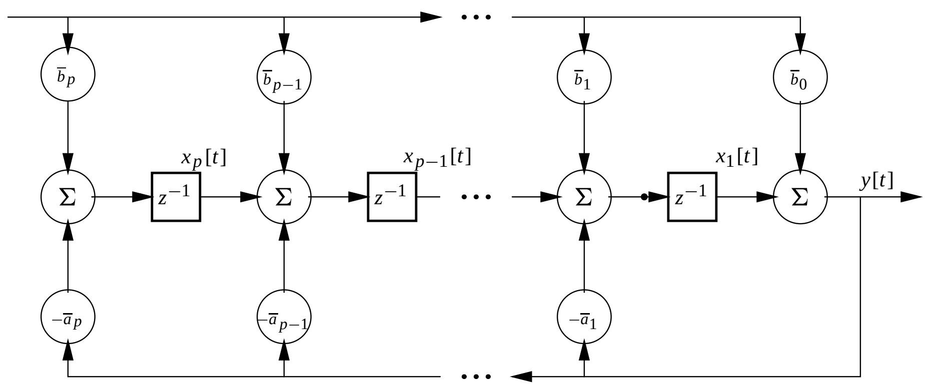
$$Y(z)\sum_{k=0}^{p} \overline{a}_k z^{-k} = \sum_{k=0}^{p} \overline{b}_k z^{-k}$$

or

$$Y(z) + \overline{a}_1 z^{-1}Y(z) + \cdots + \overline{a}_p z^{-p}Y(z) = \overline{b}_0 F(z) + \overline{b}_1 z^{-1}F(z) + \cdots + \overline{b}_p z^{-p}F(z).$$

Solving for the first $Y(z)$ on the LHS we have

$$Y(z) = \overline{b}_0 F(z) + [\overline{b}_1 F(z) - \overline{a}_1 Y(z)]z^{-1} + \cdots + [\overline{b}_p F(z) - \overline{a}_p Y(z)]z^{-p}$$



(b)

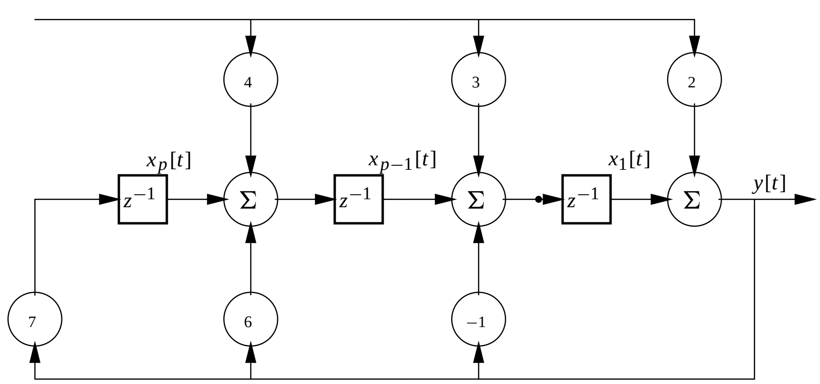
(c) Recognizing that $y[t] = x_1[t] + \bar{b}_0 f[t]$, the delay outputs can be written as

$$\begin{aligned}
x_1[t+1] &= x_2[t] + \bar{b}_1 f[t] - \bar{a}_1(x_1[t] + \bar{b}_0 f[t]) \\
x_2[t+1] &= x_3[t] + \bar{b}_2 f[t] - \bar{a}_2(x_1[t] + \bar{b}_0 f[t]) \\
&\vdots \\
x_p[t+1] &= \bar{b}_p f[t] - \bar{a}_p(x_1[t] + \bar{b}_0 f[t])
\end{aligned}$$

and the output is $y[t] = x_1[t] + \bar{b}_0 f[t]$. These equations can be expressed as

$$\begin{bmatrix} x_1[t+1] \\ x_2[t+1] \\ \vdots \\ x_p[t+1] \end{bmatrix} = \begin{bmatrix} -\bar{a}_1 & 1 & 0 & \cdots & 0 \\ -\bar{a}_2 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ -\bar{a}_{p-1} & 0 & 0 & \cdots & 1 \\ -\bar{a}_p & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1[t] \\ x_2[t] \\ \vdots \\ x_p[t] \end{bmatrix} + \begin{bmatrix} \bar{b}_1 - \bar{a}_1\bar{b}_0 \\ \bar{b}_2 - \bar{a}_2\bar{b}_0 \\ \vdots \\ \bar{b}_p - \bar{a}_p\bar{b}_0 \end{bmatrix} f[t]$$

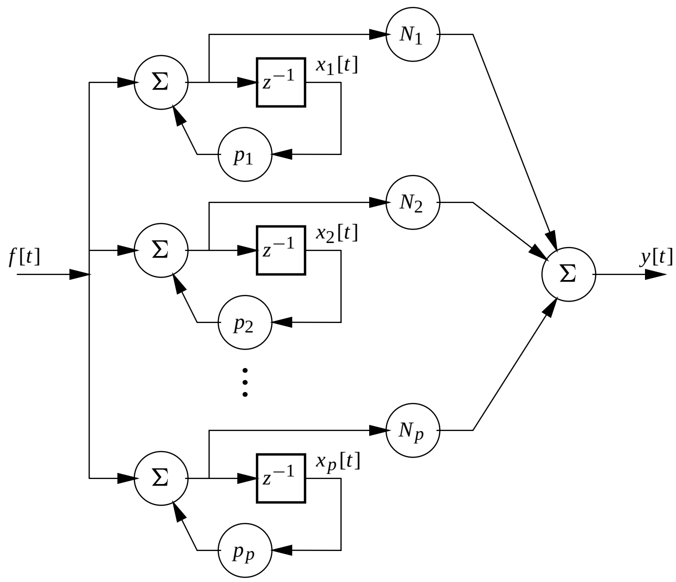
$$y[t] = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \mathbf{x}[t] + \bar{b}_0 f[t].$$



(d)

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 6 & 0 & 1 \\ 7 & 0 & 0 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 16 \\ 14 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad d = 2$$

1.4-15 Parallel form:



(a)

(b) Each state variables satisfies $x_i[t+1] = f[t] + p_i x[t]$. The output is

$$y[t] = N_1 x_1[t] + N_2 x_2[t] + \cdots + N_p x_p[t].$$

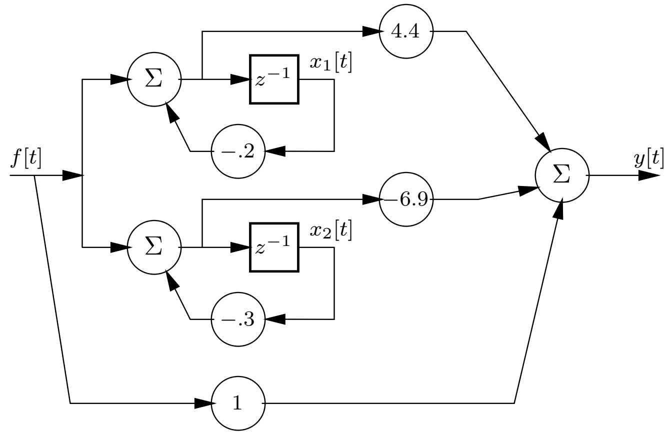
The result follows. (If $b_0 \neq 0$, then there is a direct line of weight b_0 from input to output.)

(c) PFE: Use the form usually associated with Laplace transforms:

$$H(z) = \frac{1 - 2z^{-1}}{1 + .5z^{-1} + .06z^{-2}} = \frac{(z^2 - 2z)}{(z + .2)(z + .3)} = 1 + \frac{k_1}{z + .2} + \frac{k_2}{z + .3}$$

where $k_0 = 1$ and, by CUPI,

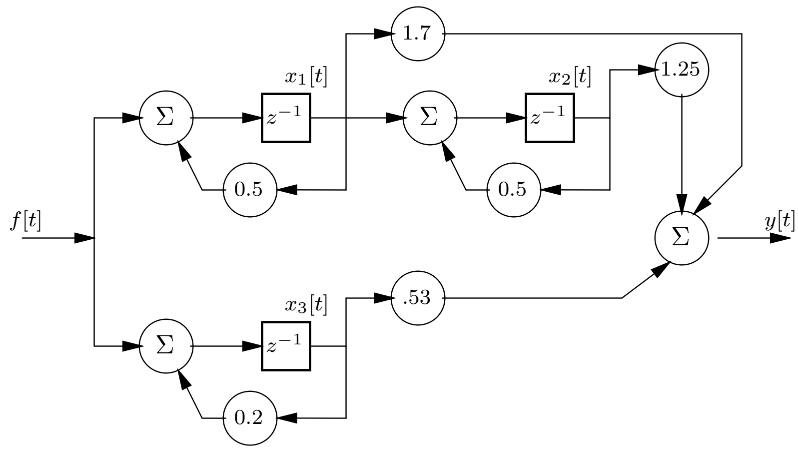
$$k_1 = 4.4 \qquad k_2 = -6.9.$$



$$A = \begin{bmatrix} -.2 & \\ & -.3 \end{bmatrix} \qquad \bar{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} -22 \\ 23 \end{bmatrix} \qquad d = 1$$

(d) Repeated roots:

$$H(z) = \frac{z^2 + 1}{(z - .2)(z - .5)} = 1 + \frac{1.6667}{z - .5} + \frac{1.25}{(z - .5)^2} + \frac{.5333}{z - .2}$$



(e)

(f) With the state assignment as shown above, the state equations are

$$x_1[t+1] = 0.5x_1[t] + f[t]$$
$$x_2[t+1] = 0.5x_2[t] + x_1[t]$$
$$x_3[t+1] = 0.2x_3[t] + f[t].$$

which has

$$A = \begin{bmatrix} .5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & .2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} 1.667 \\ 1.25 \\ .533 \end{bmatrix}$$

1.4-16 From (1.26), the transfer function of (1.21) is

$$H(z) = \frac{Y(z)}{F(z)} = (\mathbf{c}^T(zI - A)^{-1}\mathbf{b} + d).$$

Taking the Z-transform of (1.22),

$$zZ(z) = \overline{A}Z(z) + \overline{\mathbf{b}}F(z)$$
$$Y(z) = \overline{\mathbf{c}}^T Z(z) + \overline{d}F(z)$$

Solving the first for $Z(z)$, we have $Z(z) = (zI - \overline{A})^{-1}\overline{\mathbf{b}}F(z)$, which, upon substitution into the second gives

$$Y(z) = \overline{\mathbf{c}}^T(zI - \overline{A})^{-1}\overline{\mathbf{b}}F(z) + \overline{d}F(z),$$

so the new transfer function is

$$H_{\text{new}}(z) = \overline{\mathbf{c}}^T(zI - \overline{A})^{-1}\overline{\mathbf{b}} + \overline{d}$$

Now substituting

$$\overline{A} = T^{-1}AT \qquad \overline{\mathbf{b}} = T^{-1}\mathbf{b} \qquad \overline{\mathbf{c}} = T^T\mathbf{c}$$

we get

$$\overline{\mathbf{c}}^T(zI - \overline{A})^{-1}\overline{\mathbf{b}} = \mathbf{c}^T T[zI - T^{-1}AT]^{-1}T^{-1}\mathbf{b} = \mathbf{c}^T T[T^{-1}(zI - A)T]^{-1}T^{-1}\mathbf{b}$$
$$= \mathbf{c}^T TT^{-1}(zI - A)TT^{-1}\mathbf{b} = \mathbf{c}^T(zI - A)^{-1}\mathbf{b},$$

so the transfer functions are the same.

1.4-17 Solution of the state equation:

(a) (Time-invariant system) We observe that, starting from a known initial state $\mathbf{x}[0]$,

$$\mathbf{x}[1] = A\mathbf{x}[0] + \mathbf{b}f[0]$$
$$\mathbf{x}[2] = Ax[1] + \mathbf{b}f[t]$$
$$= A(A\mathbf{x}[0] + \mathbf{b}f[0]) = A^2x[0] + A\mathbf{b}f[0] + \mathbf{b}f[1]$$
$$= A^2\mathbf{x}[0] + \sum_{k=0}^{1} A^k\mathbf{b}f[2-1-k].$$

Now (the inductive hypothesis) assuming that

$$\mathbf{x}[t] = A^t\mathbf{x}[0] + \sum_{k=0}^{t-1} A^k \mathbf{b} f[t-1-k].$$

is true at time k, we have

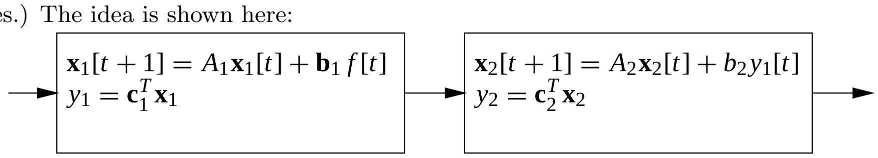
$$\begin{aligned}
\mathbf{x}[t+1] &= A\mathbf{x}[t] + \mathbf{b}f[t] \\
&= A(A^t\mathbf{x}[0] + \sum_{k=0}^{t-1} A^k\mathbf{b}f[t-1-k]) + \mathbf{b}f[t] \\
&= A^{t+1}\mathbf{x}[0] + A\sum_{k=0}^{t-1} A^{k+1}\mathbf{b}f[t-1-k] + \mathbf{b}f[t] \\
&= A^{t+1}\mathbf{x}[0] + \sum_{j=1}^{t} A^{j}\mathbf{b}f[t-j] + \mathbf{b}f[t] \qquad \text{(letting } j = k+1\text{)} \\
&= A^{t+1}\mathbf{x}[0] + \sum_{j=0}^{t} A^{j}\mathbf{b}f[t-j] + \mathbf{b}f[t]
\end{aligned}$$

(b) (Time-varying system) Proceeding as before, find

$$x[t] = \left(\prod_{i=0}^{t-1} A[i]\right)\mathbf{x}[0] + \sum_{j=0}^{t-1}\left(\prod_{i=1}^{j} A[i]\right)\mathbf{b}[t-1-j]\mathbf{f}[t-1-j]$$

1.4-18 (Interconnection of systems)

(a) (Series.) The idea is shown here:



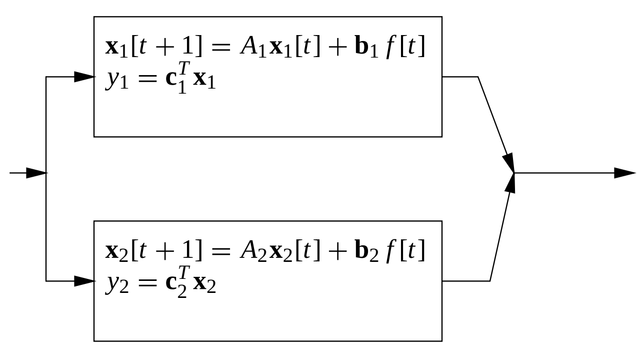
The input to the second block it $y_1[t] = \mathbf{c}_1^T\mathbf{x}_1[t]$.

$$\begin{bmatrix}\mathbf{x}_1[t+1] \\ \mathbf{x}_2[t+1]\end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ b_2\mathbf{c}_1^T & A_2\end{bmatrix}\begin{bmatrix}\mathbf{x}_1[t] \\ \mathbf{x}_2[t]\end{bmatrix} + \begin{bmatrix}\mathbf{b}_1 \\ 0\end{bmatrix} f[t]$$

$$y[t] = y_2[t] = \begin{bmatrix} 0 & \mathbf{c}_2^T\end{bmatrix}\begin{bmatrix}\mathbf{x}_1[t] \\ \mathbf{x}_2[t]\end{bmatrix}$$

so

$$A = \begin{bmatrix} A_1 & 0 \\ b_2\mathbf{c}_1^T & A_2\end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix}\mathbf{b}_1 \\ 0\end{bmatrix} \qquad \mathbf{c}^T = \begin{bmatrix} 0 & \mathbf{c}_2^T\end{bmatrix}.$$



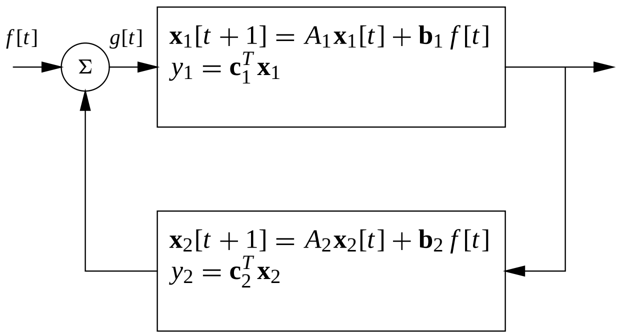
(b) (Parallel.)

$$\begin{bmatrix}\mathbf{x}_1[t+1] \\ \mathbf{x}_2[t+1]\end{bmatrix} = \begin{bmatrix} A_1 & \\ & A_2\end{bmatrix}\begin{bmatrix}\mathbf{x}_1[t] \\ \mathbf{x}_2[t]\end{bmatrix} + \begin{bmatrix}\mathbf{b}_1 \\ \mathbf{b}_2\end{bmatrix} f[t]$$

$$y[t] = \begin{bmatrix}\mathbf{c}_1^T & \mathbf{c}_2^T\end{bmatrix}\begin{bmatrix}\mathbf{x}_1[t] \\ \mathbf{x}_2[t]\end{bmatrix}$$

so

$$A = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} \qquad \mathbf{c}^T = \begin{bmatrix} \mathbf{c}_1^T & \mathbf{c}_2^T \end{bmatrix}$$



(c) (Feedback.)
Recognizing that the input to the feedback block is

$$f_2[t] = y_1 = \mathbf{c}^t\mathbf{x}_1$$

and that the input to the feedforward block is

$$g[t] = f[t] + y_2[t] = f[t] + \mathbf{c}^T\mathbf{x}_2[t],$$

the dynamics are

$$\begin{bmatrix} \mathbf{x}_1[t+1] \\ \mathbf{x}_2[t+1] \end{bmatrix} = \begin{bmatrix} A_1 & \mathbf{b}_1\mathbf{c}_2^T \\ \mathbf{b}_2\mathbf{c}_1^T & A_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1[t] \\ \mathbf{x}_2[t] \end{bmatrix} + \begin{bmatrix} \mathbf{b}_1 \\ 0 \end{bmatrix} f[t]$$

$$y[t] = y_1[t] = \begin{bmatrix} \mathbf{c}_1^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1[t] \\ \mathbf{x}_2[t] \end{bmatrix}.$$

1.4-19 The transfer function of the first system is

$$H_2(z) = \begin{bmatrix} \mathbf{c}^T & \mathbf{q}^T \end{bmatrix} \left[zI - \begin{bmatrix} A & A_1 \\ 0 & A_2 \end{bmatrix} \right]^{-1} \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix}$$

Then

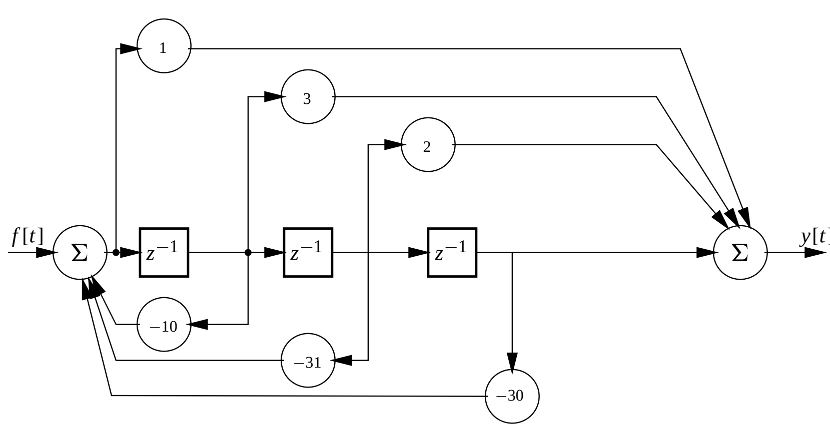
$$\left[zI - \begin{bmatrix} A & A_1 \\ 0 & A_2 \end{bmatrix} \right]^{-1} = \begin{bmatrix} zI - A & -A_1 \\ 0 & zI - A_2 \end{bmatrix}^{-1} = \begin{bmatrix} (zI-A)^{-1} & (zI-A)^{-1}A_1(zI-A_2)^{-1} \\ 0 & (zI-A_2)^{-1} \end{bmatrix}$$

(See section 4.12 on inverses of block matrices.) Then

$$H_2(z) = \begin{bmatrix} \mathbf{c}^T & \mathbf{q}^T \end{bmatrix} \begin{bmatrix} (zI-A)^{-1} & (zI-A)^{-1}A_1(zI-A_2)^{-1} \\ 0 & (zI-A_2)^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{c}^T & \mathbf{q}^T \end{bmatrix} \begin{bmatrix} (zI-A)^{-1}\mathbf{b} \\ 0 \end{bmatrix} = \mathbf{c}^T(zI-A)^{-1}\mathbf{b}$$

which is the transfer function as the system $(A, \mathbf{b}, \mathbf{c}^T)$. Computations are similar for the second system. Since A_1 or A_2 can take on various dimensions, the dimension of the state variable is not necessarily the same as the degree of the final transfer function.

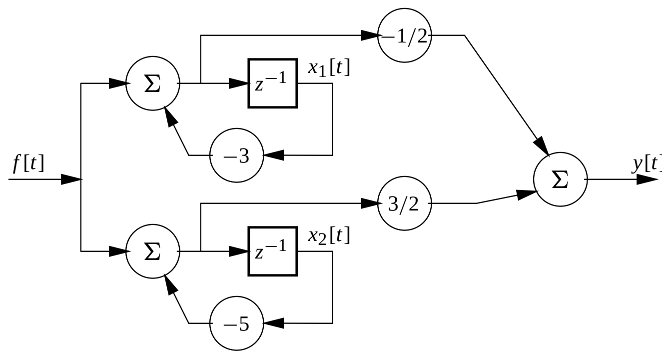
1.4-20 Realizations



(a)

(b) After factoring, we see that

$$H(z) = \frac{z(z+3)(z+2)(z+1)}{(z+5)(z+3)(z+1)} = \frac{z(z+2)}{(z+3)(z+5)} = \frac{(-1/2)}{1+3z^{-1}} + \frac{(3/2)}{1+5z^{-1}}$$



(c) There are two modes in a minimal realization.

1.4-21 From the system

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x} + \mathbf{b}f(t)\\ y(t) &= \mathbf{c}^T\mathbf{x}(t) + df(t),\end{aligned}$$

solve the second for $f(t)$ to obtain

$$f(t) = \frac{1}{d}y(t) - \frac{\mathbf{c}^T}{d}\mathbf{x}(t). \tag{1.1}$$

Substituting into the first and re-arranging we find

$$\dot{\mathbf{x}}(t) = (A - \frac{\mathbf{b}\mathbf{c}^t}{d})\mathbf{x}(t) + \frac{\mathbf{b}}{d}\mathbf{y}(t)$$

If we now interpret this as a system with input $y(t)$, and output equation given by (1.1), we note that the system has the form

$$(A - \frac{\mathbf{b}\mathbf{c}^t}{d}, \frac{\mathbf{b}}{d}, -\frac{\mathbf{c}^T}{d}, \frac{1}{d}).$$

1.4-22 State-space solutions:

(a) Starting from

$$\mathbf{x}(t) = e^{A(t-\tau)}\mathbf{x}(\tau) + \int_\tau^t e^{A(t-\tau)}B\mathbf{u}(\lambda)\,d\lambda,$$

we have, using the properties of the matrix exponential,

$$\begin{aligned}\dot{\mathbf{x}}(t) &= Ae^{A(t-\tau)}\mathbf{x}(\tau) + \int_\tau^t Ae^{A(t-\tau)}B\mathbf{u}(\lambda)\,d\lambda + e^{A(t-t)}B\mathbf{f}(\lambda)\\ &= A\mathbf{x}(t) + B\mathbf{f}(\lambda)\end{aligned}$$

(b) We have

$$\mathbf{x}(t) = \Phi(t,0)\mathbf{x}(0) + \int_0^t \Phi(t,\lambda)B(\lambda)\mathbf{f}(\lambda)d\lambda$$

so that

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A(t)\Phi(t,0)\mathbf{x}(0) + \int_0^t A(t)\Phi(t,\lambda)B(\lambda)\mathbf{f}(\lambda)\,d\lambda + \Phi(t,t)B(t)\mathbf{f}(t)\\ &= A(t)\mathbf{x}(t) + B(t)\mathbf{f}(t).\end{aligned}$$

1.4-23 It is straightforward to verify that

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t)\\ x_2(t)\end{bmatrix} = \begin{bmatrix} C\sin(t+\theta)\\ C\cos(t+\theta)\end{bmatrix}$$

satisfies the differential equation. The constants C and θ are chosen to match the initial conditions $\mathbf{x}(0) = \mathbf{x}_0$.

1.4-24

$$H(s) = \frac{-2s}{s^2+s-2} = \frac{-(4/3)}{s+2} + \frac{-(2/3)}{s-1}.$$

The modes occur at $s = -2$ and $s = 1$, and A has the eigenvalues -2 and 1.

1.4-25 Simply perform the division $\frac{1}{1-x}$.

1.4-26 (System identification)

(a)

$$H_c(s) = \frac{b/(s(s+a))}{1+b/(s(s+a))} = \frac{b}{s(s+a)+b} = \frac{1}{1+(a/b)s+(1/b)s^2}.$$

(b) We have

$$H_c(j\omega) = \frac{1}{(1-\omega^2/b)+(a\omega/b)j}$$

so

$$|H_c(j\omega)|^2 = \frac{1}{(1-\omega^2/b)^2+(a\omega/b)^2} = \frac{1}{\frac{1}{b^2}[(b-\omega^2)^2+(a\omega)^2]}$$

from which $A(j\omega)$ follows. Also

$$\tan\arg H_c(j\omega) = \frac{-a\omega/b}{1-\omega^2/b^2} = -\frac{a\omega}{b-\omega^2}$$

so that $\tan\arg\frac{1}{H_c(j\omega)} = \frac{a\omega}{b-\omega^2}$.

(c) From the equations of the previous part, we have

$$\begin{aligned} A(j\omega)b &= \sqrt{(b-\omega^2)^2+(a\omega)^2} \\ &= \sqrt{(a\omega/\tan\phi(j\omega))^2+(a\omega)^2} \\ &= a\omega\sqrt{1/(\tan\phi(j\omega))^2+1}. \end{aligned}$$

We recognize an equation of the form

$$\begin{bmatrix} A(j\omega) & -\omega\sqrt{1/(\tan\phi(j\omega))^2+1} \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} = 0.$$

From $\tan\phi(j\omega) = \frac{a\omega}{b-\omega^2}$ we also obtain the equation

$$b\tan\phi(j\omega) - \omega^2\tan\phi(j\omega) = a\omega,$$

from which we recognize an equation of the form

$$\begin{bmatrix} \tan\phi(j\omega) & -\omega \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} = \omega^2\tan\phi(j\omega).$$

Since these are true for any ω, we make measurements at several frequencies, and stack up the equations to obtain the over-determined set of equations.

1.4-27

$$\begin{aligned} G_{yy}(\omega) &= |Y(\omega)|^2 = Y(\omega)\overline{Y}(\omega) \\ &= \sum_l y[l]e^{-j\omega l}\sum_k \overline{y}[k]e^{j\omega k} = \sum_k\sum_l y[l]\overline{y}[k]e^{-j\omega(l-k)} \\ &= \sum_k \rho_{yy}[l-k]e^{-j\omega(l-k)} \qquad \forall l \\ &= \sum_k \rho_{yy}[k]e^{-j\omega k}. \end{aligned}$$

1.4-28

$$\begin{aligned} \sum_t |y[t]|^2 &= \sum_t y[t]\frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{Y}(\omega)e^{-j\omega t}\,d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{Y}(\omega)\sum_t y[t]e^{-j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{Y}(\omega)Y(\omega)\,d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}G_{yy}(\omega)\,d\omega. \end{aligned}$$

1.4-29 First the hint:

$$\sum_{n=1}^{N}\sum_{m=1}^{n} f(n-m) = \sum_{n=m} f(0) + \sum_{n-m=-1} f(-1) + \sum_{n-m=1} f(1) + \cdots + \sum_{n-m=-(N-1)} f(-(N-1)) + \sum_{n-m=N-1} f(N-1)$$

The first sum on the right has N terms, the next two sums have $N-1$ terms, and so on to the last two sums, which have only one term each. This can written as

$$\sum_{n=1}^{N}\sum_{m=1}^{N} f(n-m) = \sum_{l=-N+1}^{N-1} (N-|l|)f(l).$$

Now for the problem:

$$\begin{aligned}\lim_{N\to\infty} E\left[\frac{1}{N}\left|\sum_{n=1}^{N} y[n]e^{-j\omega n}\right|^2\right] &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\sum_{m=1}^{N} E[y[n]\overline{y}[m]]e^{-j\omega(n-m)} \\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{l=1-N}^{N-1}(N-|l|)r_{yy}[l]e^{-j\omega l} \\ &= \sum_{l=-\infty}^{\infty} r_{yy}[l]e^{-j\omega l} - \lim_{N\to\infty}\frac{1}{N}\sum_{l=1-N}^{N-1}|l|r_{yy}[l]e^{-j\omega l}.\end{aligned}$$

Under the assumption (1.39), the second sum vanishes in the limit, leaving the PSD.

1.4-30 (Modal analysis)

(a) Since the data are given as third-order, the following set of equations can be used to find the coefficients of the system:

$$\begin{bmatrix} -0.1 & -0.25 & -0.32 \\ 0.0222 & -0.1 & -0.25 \\ -0.0006 & 0.0222 & -0.1 \\ 0.0012 & -0.0006 & 0.0222 \\ -0.0005 & 0.0012 & -0.0006 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -0.0222 \\ 0.0006 \\ -0.0012 \\ 0.0005 \\ -0.0001 \end{bmatrix}$$

from which the solution is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.175523 \\ 0.00351197 \\ 0.0117816 \end{bmatrix},$$

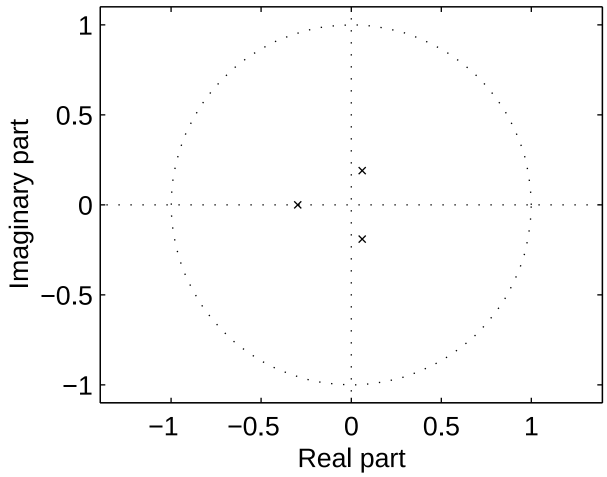
corresponding to the system equation

$$y[t+3] + (.1755)y[t+2] + (.00351197)y[t+1] + (.0117816)y[t] = 0.$$

The roots are at

$$[p_1, p_2, p_2] = [-.297, 0.0608 \pm j.1896],$$

as shown by the Z-plane plot.



(b) An equation to find the coefficients is

$$\begin{bmatrix} 1 & 1 & 1 \\ p_1 & p_2 & p_3 \\ p_1^2 & p_2^2 & p_3^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} y[0] \\ y[1] \\ y[2] \end{bmatrix}$$

from which

$$[c_1, c_2, c_3] = \begin{bmatrix} 0.5015 & -0.0907 - 1.0813j & -0.0907 + 1.0813j \end{bmatrix}$$

1.4-31 By a trigonometric identity,

$$y[t] = A\cos\omega_1 t\cos\theta_1 + A\sin\omega_1 t\sin\theta_1 + B\cos\omega_2 t\cos\theta_2 + B\sin\omega_2 t\sin\theta_2$$

Identify $x_1 = A\cos\theta_1$, $y_1 = A\sin\theta_1$, $x_2 = B\cos\theta_2$ and $y_2 = B\sin\theta_2$ as unknowns. Then from measurements at $t = t_1, t_2, \ldots, t_N$ the following equations can be obtained:

$$\begin{bmatrix} \cos\omega_1 t_1 & \sin\omega_1 t_1 & \cos\omega_2 t_1 & \sin\omega_2 t_1 \\ \cos\omega_1 t_2 & \sin\omega_1 t_2 & \cos\omega_2 t_2 & \sin\omega_2 t_2 \\ \vdots & & & \\ \cos\omega_1 t_N & \sin\omega_1 t_N & \cos\omega_2 t_N & \sin\omega_2 t_N \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} y[t_1] \\ y[t_2] \\ \vdots \\ y[t_N] \end{bmatrix}.$$

By solving this over-determined set of equations, determine x_1, x_2, y_1 and y_2, after which

$$A = \sqrt{x_1^2 + y_1^2} \qquad B = \sqrt{x_2^2 + y_2^2}$$
$$\theta_1 = \tan^{-1}(y_1/x_1) \qquad \theta_2 = \tan^{-1}(y_2/x_2)$$

1.6-32 Simply multiply and simplify, and show that the product is the identity:

$$R^{-1}R = \frac{1}{1-\rho^2}\begin{bmatrix} \frac{1}{\sigma_1^2} & \frac{-\rho}{\sigma_1\sigma_2} \\ \frac{-\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2} \end{bmatrix} \cdot \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

1.6-33 For notational convenience, we will deal with the case when $\mu_1 = \mu_2 = 0$. In the exponent of (1.47) we have

$$\begin{aligned} -\frac{1}{2}\mathbf{w}^T R^{-1}\mathbf{w} &= -\frac{1}{2(1-\rho^2)}\begin{bmatrix} w_1 & w_2 \end{bmatrix}\begin{bmatrix} \frac{1}{\sigma_1^2} & \frac{-\rho}{\sigma_1\sigma_2} \\ \frac{-\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2} \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\ &= -\frac{1}{2(1-\rho^2)}\left[\frac{w_1^2}{\sigma_1^2} - \frac{2\rho w_1 w_2}{\sigma_1\sigma_2} + \frac{w_2^2}{\sigma_2^2}\right]. \end{aligned}$$

In the coefficient in front we have

$$|R|^{1/2}| = \sqrt{\sigma_1^2\sigma_2^2 - \sigma_{12}^2} = \sigma_1\sigma_2\sqrt{1 - \sigma_{12}^2/(\sigma_1\sigma_2)} = \sigma_1\sigma_2\sqrt{1-\rho^2}.$$

These lead to (1.51).

1.6-34 (a) $Y \sim \mathcal{N}(\mu_x, \sigma_x^2 + \sigma_n^2)$

(b) The correlation coefficient is

$$\rho = \frac{\sigma_x}{\sqrt{\sigma_x^2 + \sigma_y^2}}.$$

From (1.53), an estimate of X given the measurement $Y = y$ is

$$\hat{x} = \mu_x + \frac{\sigma_x}{\sigma_y}\rho(y - \mu_y) = \mu_x + \frac{\sigma_x}{\sqrt{\sigma_x^2 + \sigma_n^2}}\rho(y - \mu_y) = \mu_x + \frac{\sigma_x^2}{\sigma_x^y + \sigma_y^2}.$$

When $\sigma_n^2 \gg \sigma_x^2$, then we have approximately

$$\hat{x} \approx \mu_x,$$

and the variance is $\approx \sigma_x^2$. In this case, when the noise is too high, the best thing to do is ignore the measurement and use the prior knowledge about X, and we are hardly less certain about the variance than we were before the measurement. When $\sigma_x^2 \gg \sigma_n^2$, then we have approximately

$$\hat{x} \approx \mu_u + \rho(y - \mu_y) \approx y + \mu_x - \mu_y,$$

so that we linearly adjust the mean be the difference between the measured value and the expected value of Y. The variance is approximately the variance in the noise.

1.6-35 $Z \sim \mathcal{N}(a\mu_x + b\mu_y, a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\rho\sigma_x\sigma_y)$.

1.6-36 $E[Y] = E[\sigma X + \mu] = \mu$. To determine the variance, it is most convenient to assume $\mu = 0$. Then

$$\text{var}(Y) = E[(\sigma X)^2] = \sigma^2.$$

1.6-37 (a) ML estimate: Maximizing $f(x_1, x_2, \ldots, x_n|\mu, \sigma^2)$ is equivalent to maximizing $\log f(x_1, x_2, \ldots, x_n|\mu, \sigma^2)$, since log is an increasing function. Taking the derivative with respect to μ of $\log f(x_1, x_2, \ldots, x_n|\mu, \sigma^2)$ and equating to zero we obtain

$$\frac{\partial}{\partial \mu}\log f(x_1, x_2, \ldots, x_n|\mu, \sigma^2) = -\frac{2}{2\sigma^2}\sum_{i=1}^n (x_i - \mu) = 0.$$

Solving this for μ, and calling the result $\hat{\mu}$ — the estimate of μ, we obtain

$$\hat{\mu} = \frac{1}{n}x_i.$$

(b) $E\hat{\mu} = \mu$. The estimate is *unbiased*.

(c)

$$\begin{aligned}\text{var}(\hat{\mu}) &= E[\hat{\mu}^2] - E[\hat{\mu}]^2 = E[\frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n x_i x_j] - \mu^2 \\ &= \frac{1}{n^2}\left[\sum_{i=1}^n E[x_i^2] + \sum_{i\neq j} E[x_i x_j]\right] - \mu^2 = \frac{1}{n^2}\left[n(\sigma^2 + \mu^2) + (n^2 - n)\mu^u\right] - \mu^2 \\ &= \frac{\sigma^2}{n}.\end{aligned}$$

The variance of the estimate of the mean goes down as $1/n$.

(d) Taking the derivative of $\ln f$ and equating to zero we have

$$\begin{aligned}\frac{\partial}{\partial \sigma^2}\ln f &= \frac{\partial}{\partial \sigma^2} - \frac{n}{2}\ln 2\pi - \frac{n}{2}\ln \sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \mu)^2 \\ &= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^n (x_i - \mu)^2 = 0.\end{aligned}$$

Denoting the value of σ^2 which solves this as $\hat{\sigma}^2$, we have

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \mu)^2$$

1.7-38 (a) $f(x_3, x_1|x_2) = f(x_3|x_1, x_2)f(x_1|x_2) = f(x_3|x_2, f(x_1|x_2)$

(b)

$$\begin{aligned} r_x(t_3, t_1) &= E[X(t_3)X(t_1)] = E[E[X(t_3)X(t_1)|X(t_2)]] \\ &= E[E[X(t_3)|X(t_2)]E[X(t_1)|X(t_2)]] = E[(\rho_{2,3}\frac{\sigma_3}{\sigma_2}X_2)(\rho_{1,2}\frac{\sigma_1}{\sigma_2}X_2)] \\ &= \frac{\rho_{2,3}\rho_{1,2}\sigma_1\sigma_2^2\sigma_3}{\sigma_2^2} = \frac{r_x(t_3, t_2)r_x(t_2, t_1)}{r_x(t_2, t_2)}. \end{aligned}$$

1.7-39 If $A\mathbf{p} = \mathbf{p}$, then $\mathbf{p}$ must be the eigenvector of A corresponding to the eigenvalue 1, scaled to have the sum of its elements equal to 1. The vector

$$\mathbf{p} = \begin{bmatrix} 0.571988 \\ 0.544086 \\ 0.613841 \end{bmatrix},$$

obtained using the `eig` function of MATLAB is the solution.

1.8-40 Assume to the contrary that $\sqrt{3}$ is rational,

$$\sqrt{3} = \frac{n}{m},$$

where the fraction is expressed in reduced terms. Then $3m^2 = n^2$, so n^2 must be a multiple of three. This implies that n must be a multiple of three, for if $n = 2k + 1$ or $n = 2k + 2$, then n^2 is not a multiple of three. We thus obtain

$$3m^2 = (3k)^2 = 9k^2,$$

or $3k^2 = m^2$, or

$$\sqrt{3} = \frac{m}{k}.$$

But now the numerator and denominator are smaller than before, which contradicts our assumption the fraction n/m is in reduced form. (Alternatively, we could observe that the process could be iterated an indefinite number of times, reaching an absurd conclusion. This is the *reductio ad absurdium* argument.)

1.8-41 Suppose to the contrary that there are a finite number n of primes in the list $P = \{2, 3, 5, \ldots, p_n\}$. Form the number

$$m = 2 \cdot 3 \cdot 5 \cdot \cdots \cdot p_1 + 1$$

(the product of all the presumed primes, plus 1). Then m is not divisible by any of the primes in the set P thus m is either a product of primes, or it is prime. If m is prime, then since m is larger than any of the numbers in the original list we have found a new prime, contradicting the assumption that there are a finite number of primes.

If m is the product of two primes, write $m = qr$, where q is prime. Since none of the numbers in P divides m, but q does, it must be a new prime, again yielding a contradiction.

1.8-42 If $m = 2k + 1$ for some integer k (i.e., m is odd) then

$$m^2 = 4k^2 + 4k + 1,$$

an odd number. If $m = 2k$, then

$$m^2 = 4k^2,$$

an even number.

1.8-43 It seems plausible that

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1.$$

This is easily verified for small n. Taking this now as an inductive hypothesis under n, we try it for $n + 1$:

$$\begin{aligned} \sum_{i=0}^{n+1} 2^i &= \sum_{i=0}^{n} 2^i + 2^{n+1} = (2^{n+1} - 1) + 2^{n+1} \\ &= 2(2^{n+1}) - 1 = 2^{n+2} - 1. \end{aligned}$$

1.8-44 Plausible rule:

$$\sum_{i=1}^{n} (2i - 1) = n^2.$$

This may be easily verified for small n. Taking this as the inductive hypothesis for n, we try it for $n+1$:

$$\sum_{i=1}^{n+1}(2i-1) = \sum_{i=1}^{n}(2i-1) + 2(n+1) - 1$$
$$= n^2 + 2(n+1) - 1 = n^2 + 2n + 1 = (n+1)^2.$$

1.8-45 With some experimentation, it appears that

$$\sum_{i=1}^{n} \frac{1}{i^2+i} = \frac{n}{n+1}.$$

Taking this as the inductive hypothesis, we find for $n+1$

$$\sum_{i=1}^{n+1} \frac{1}{i^2+i} = \sum_{i=1}^{n} \frac{1}{i^2+i} + \frac{1}{(n+1)^2+n+1} = \frac{n}{n+1} + \frac{1}{n^2+3n+2}$$
$$= \frac{n+1}{n+2}.$$

1.8-46 The result is immediate for $n = 1$. Assume true for n. Then

$$(n+1)^3 - (n+1) = n^3 + 3n^2 + 2n = (n^3 - n) + (3n^2 + 3n).$$

By hypothesis, $3|n^3 - n$, and it is clear that $3|(3n^2 + 3n)$, so 3 must divide their sum.

1.8-47 The result is straightforward to verify for small n. Taking the stated result as the inductive hypothesis, we have

$$\binom{n+1}{k} + \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} + \binom{n+1}{k-1}$$
$$= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} + \frac{(n+1)!}{(k-1)!(n-k+2)!}$$
$$= \frac{n!(n-k+1)(n-k+2) + kn!(n-k+2) + k(n+1)!}{k!(n-k+2)!} = \frac{n!(n^2+2n+2)}{k!(n-k+2)!}$$
$$= \frac{(n+2)!}{k!(n-k+2)!}$$
$$= \binom{n+2}{k}.$$

1.8-48 The result is clear when $n = 0$ and $n = 1$. Taking

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

as the inductive hypothesis, we have

$$\sum_{k=0}^{n+1} \binom{n+1}{k} = \sum_{k=0}^{n} \binom{n+1}{k} + \binom{n+1}{n+1}$$
$$= \sum_{k=1}^{n} \binom{n+1}{k} + \binom{n+1}{0} + 1$$
$$= \sum_{k=1}^{n} \binom{n}{k} + \binom{n}{k-1} + 1 + 1 \qquad \text{(using the previous problem)}$$
$$= (2^n - 1) + \sum_{j=0}^{n-1} \binom{n}{j} + 1 + 1 \qquad \text{(using the inductive hypothesis)}$$
$$= (2^n - 1) + (2^n - \binom{n}{n}) + 1 + 1$$
$$= 2^n + 2^n = 2^{n+1}.$$

Another approach is to using the binomial theorem in the next problem, with $x = y = 1$.

1.8-49 Taking the binomial theorem as the inductive hypothesis, we have

$$\begin{aligned}
(x+y)^{n+1} &= (x+y)(x+y)^n = (x+y)\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k} \\
&= \sum_{k=0}^{n}\binom{n}{k}x^{k+1}y^{n-k} + \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k+1} \\
&= \sum_{j=1}^{n+1}\binom{n}{j-1}x^j y^{n-j+1} + \sum_{k=1}^{n}\binom{n}{k}x^k y^{n-k+1} + y^{n+1} \qquad \text{(let } j=k+1 \text{ in the first sum)} \\
&= \sum_{k=1}^{n+1}\left[\binom{n}{k-1}+\binom{n}{k}\right]x^k y^{n-k+1} + y^{n+1} \\
&= \sum_{k=1}^{n+1}\binom{n+1}{k}x^k y^{n-k+1} + y^{n+1} \qquad \text{(using (1.82))} \\
&= \sum_{k=0}^{n+1}\binom{n+1}{k}x^k y^{n-k+1}.
\end{aligned}$$

1.8-50 Taking the sum as the inductive hypothesis, we have

$$\begin{aligned}
\sum_{k=1}^{n+1}k^2 = \sum_{k=1}^{n}k^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\
&= \frac{2n^3+9n^2+13n+6}{6} = \frac{(n+1)(n+2)(2(n+1)+1)}{6}.
\end{aligned}$$

1.8-51

$$\sum_{k=1}^{n+1}r^k = \sum_{k=1}^{n}r^k + r^{n+1} = \frac{r^{n+1}-1}{r-1} + r^{n+1} = \frac{r^{n+2}-1}{r-1}.$$

1.8-52 Let $p_n = \prod_{i=1}^{n}\frac{2i-1}{2i}$. Then the inductive hypothesis can be stated as

$$\frac{1}{\sqrt{4n+1}} < p_n \leq \frac{1}{\sqrt{3n+1}}.$$

For the inductive step, we must show that

$$\frac{1}{\sqrt{4(n+1)+1}} < p_n\frac{2(n+1)-1}{2(n+1)} < \frac{1}{\sqrt{3(n+1)+1}}$$

The inequality on the left is established by using the fact that under the inductive hypothesis

$$\frac{1}{\sqrt{4n+1}} < p_n.$$

Then we must show that

$$\frac{1}{\sqrt{4n+5}} < \frac{1}{\sqrt{4n+1}}\frac{2n+1}{2n+2}.$$

Cross-multiplying and squaring, this is equivalent to showing that

$$(2n+1)^2(4n+5) > (2n+2)^2(4n+1).$$

Expanding both sides, this is equivalent to showing that

$$16n^3+36n^2+24n+5 > 16n^3+36n^2+24n+4$$

which is equivalent to $1>0$. Hence the left inequality is established.
By the inductive hypothesis,

$$p_n \leq \frac{1}{\sqrt{3n+1}}.$$

To establish the right inequality, we must show that

$$\frac{1}{\sqrt{3n+1}}\frac{2n+1}{2n+2} \le \frac{1}{\sqrt{3n+4}}$$

Squaring and cross-multiplying, this is equivalent to showing that

$$(3n+4)(2n+1)^2 < (3n+1)(2n+2)^2,$$

Which is equivalent to showing that

$$4n^2 + 5n + 1 > 0,$$

which is true for any positive n.

1.8-53 This is clearly true for $n = 1$. Assuming true for n, consider $x^{n+1} - y^{n-1}$. By long division it is straightforward to show that

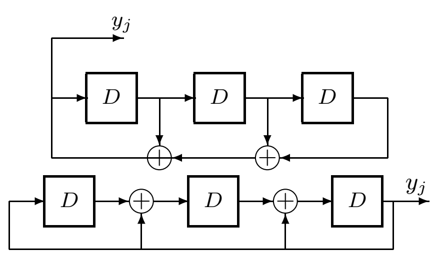
$$\frac{x^{n+1} - y^{n+1}}{x - y} = x^n + y^n + \frac{yx^n - xy^n}{x - y}$$
$$= x^n + y^n + \frac{xy(x^{n-1} - y^{n-1})}{x - y}$$

By the inductive hypothesis, the quotient on the right divides with no remainder, so the result is proved.

1.9-54

j	state	y_j (output)
0	0001	1
1	1000	1
2	1100	0
3	0110	0
4	0011	0
5	0001	1
⋮	⋮	⋮

1.9-55



(a)

(b) For the first realization,

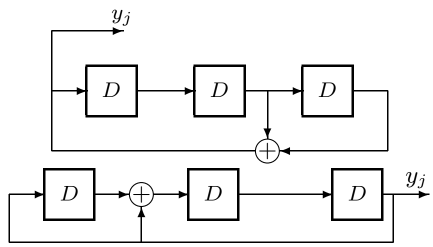
j	state	y_j (output)
0	001	1
1	100	1
2	110	0
3	011	0
4	001	1
⋮	⋮	⋮

For the second realization,

j	state	y_j (output)
0	001	1
1	111	1
2	100	0
3	010	0
4	001	1
⋮	⋮	⋮

There are four states.

1.9-56



(a)

(b) First realization:

j	state	y_j (output)
0	001	1
1	100	0
2	010	1
3	101	1
4	110	1
5	111	0
6	011	0
7	001	1
⋮	⋮	⋮

Second realization:

j	state	y_j (output)
0	001	1
1	110	0
2	011	1
3	111	1
4	101	1
5	100	0
6	010	0
7	001	1
⋮	⋮	⋮

1.9-57 Massey:

(a) Hand computations:

Initialize: $L = 0, c = 1, p = 1, d_m = 1$

$n = 0$ $d = 0$, so $s = 2$

$n = 1$ $d = 0$, so $s = 3$

$n = 2$ $d = 0$, so $s = 4$

$n = 3$ $d = 1$

Update with length change:

$t = 1$

$c = 1 + D^4$

$p = 1$

$d_m = 1$

$s = 1$

$L = 4$

$n = 4$ $d = 0$, so $s = 2$

$n = 5$ $d = 1$

Update with no length change:

$t = 1 + D^4$

$c = 1 + D^4 + D^2$

$s = 3$

$n = 6$ $d = 0$, so $s = 4$

(b) The MATLAB operations can be summarized as

```
  y = [0 0 0 1 0 1 0]
  c = massey(y)
c = [1 0 1 0 1]
```

1.9-58

1.9-59 LFSR and polynomial division:

(a) Given that the output is known from time y_0, by (1.63) we have

$$\sum_{i=0}^{p} c_i y_{j-i} = 0 \qquad \forall j \geq p. \tag{1.2}$$

Now with $C(D) = 1 + c_1 D + \cdots + c_p D^p$ and $Y(D) = y_0 + y_1 D + \cdots$, the product is

$$C(D)Y(D) = y_0 + D(c_1 y_0 + y_1 c_0) + \cdots + D^{p-1} \sum_{i=0}^{p-1} c_i y_{p-1-i} + D^p \sum_{i=0}^{p} c_i y_{p-i} + D^{p+1} \sum_{i=0}^{p} c_i y_{p+1-i} + \cdots$$

By (1.2), all the coefficients from D^p upward are zero. We thus have

$$\begin{aligned} Z(D) &= y_0 + D(c_1 y_0 + y_1 c_0) + \cdots + D^{p-1} \sum_{i=0}^{p-1} c_i y_{p-1-i} \\ &= z_0 + z_1 D + \cdots + z_{p-1} D^p. \end{aligned}$$

(b) Equating coefficients, we have

$$\begin{aligned} z_0 &= y_0 \\ z_1 &= c_1 y_0 + c_0 y_1 \\ z_2 &= c_2 y_0 + c_1 y_1 + c_0 y_2 \\ &\vdots \\ z_{p-1} &= c_{p-1} y_0 + c_{p-2} y_1 + \cdots + c_0 y_{p-1}, \end{aligned}$$

which leads to the indicated matrix equation.

1.9-60 The $Z(D)$ coefficients can be determined from

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 = y_0 \\ 0 = y_1 \\ 0 = y_2 \end{bmatrix} = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Then by direct long division,

$$Y(D) = 1 + D^3 + D^5 + D^6 + D^7 + D^{10} + D^{12} + D^{13} + \cdots$$

which corresponds to the series

$$\{1, 0, 0, 1, 0, 1, 1, 1, 0, 0, 1, 0, 1, 1, \ldots\}.$$

It is straightforward to verify that this is the output of an LFSR with coefficients $C(D)$ and initial conditions $\{0, 0, 1\}$.

Incidentally, the long division can be computed easily in MATHEMATICA using a command such as

```
PolynomialMod[Normal[Series[(1+d^2)/(1+d^2+d^3),{d,0,15}]],2]
```

which provides up to the d^{14} term.

1.9-61 The sequence is $\{1, 0, 0, 1, 1, 1, 0\}$. We recognize that the cyclic autocorrelation is the cyclic convolution of $y[t]$ and $y[-t]$, and may be computed using an FFT. MATLAB code for this purpose, and the results of the computation, are shown below.

Algorithm 1.2 Cyclic autocorrelation

```
function plotlfsrautoc(y)
% plot the autocorrelation of the output of an LFSR

N = length(y);
yf = fft(y);
yrf = fft(y(N:-1:1)); % time-reversed
zf = yf .* yrf; % multiply
z = real(ifft(zf));
% now shift, so the zero lag is in the middle
x = [z(N- (N-1)/2:N) z(1:(N-1)/2)]
subplot(2,2,1);
plot(0:N-1, x)
set(gca,'XTickLabel',[-(N-1)/2:(N-1)/2]);
xlabel('lag')
ylabel('cyclic autocorrelation')
```