SECOND EDITION

ADVANCED ENGINEERING ELECTROMAGNETICS

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SOLUTIONS MANUAL

1.1 $\nabla \times \underline{H} = \underline{J}_{ic} + \frac{\partial \underline{D}}{\partial t}$

Taking the divergence of both sides

$$\nabla \cdot (\nabla \times \underline{H}) = \nabla \cdot \underline{J}_{ic} + \nabla \cdot \frac{\partial \underline{D}}{\partial t} = \nabla \cdot \underline{J}_{ic} + \frac{\partial}{\partial t} \nabla \cdot \underline{D}$$

Using the vector identity of

$\nabla \cdot (\nabla \times \underline{A} = 0)$ and (1-3) we can write that

$$0 = \nabla \cdot \underline{J}_{ic} + \frac{\partial}{\partial t}(q_{ve}) \Rightarrow \boxed{\nabla \cdot \underline{J}_{ic} = -\frac{\partial q_{ve}}{\partial t}}$$

1.2 $\nabla \times \underline{E} = -\underline{M}_i - \frac{\partial \underline{B}}{\partial t}$

Taking a surface integral of both sides, we can write that

$$\iint_S (\nabla \times \underline{E}) \cdot d\underline{s} = -\iint_S \underline{M}_i \cdot d\underline{s} - \frac{\partial}{\partial t} \iint_S \underline{B} \cdot d\underline{s}$$

Applying Stokes' theorem of (1-7) to the left side of the equation above leads to

$$\oint_C \underline{E} \cdot d\underline{\ell} = -\iint_S \underline{M}_i \cdot d\underline{s} - \frac{\partial}{\partial t} \iint_S \underline{B} \cdot d\underline{s}$$

Using the same procedure, we can write that

$$\oint_C \underline{H} \cdot d\underline{\ell} = \iint_S \underline{J}_i \cdot d\underline{s} + \iint_S \underline{J}_c \cdot d\underline{s} + \frac{\partial}{\partial t} \iint_S \underline{D} \cdot d\underline{s}$$

For the remaining three equations of Table 1-1 we proceed as follows:

$$\nabla \cdot \underline{D} = q_{ve}$$

Taking a volume integral of both sides, we can write that

$$\iiint_V \nabla \cdot \underline{D} \, dv = \iiint_V q_{ve} ds = Q_e$$

Applying the divergence theorem of (1-8) on the left side of the equation above leads to

$$\oiint_S \underline{D} \cdot d\underline{s} = Q_e$$

Using the same procedure, we can write that

$$\oiint_S \underline{B} \cdot d\underline{s} = Q_m$$

and

$$\oiint_S \underline{J}_{ic} \cdot d\underline{s} = -\frac{\partial}{\partial t}\iiint_V q_{ve}\, dv = -\frac{\partial Q_e}{\partial t}$$

1.3

(a) $\underline{D} = \hat{a}_x(3+x)$

$$Q_e = \oiint_S \underline{D} \cdot d\underline{s} = \int_0^1\int_0^1 \hat{a}_x(3+x)\Big|_{x=0} \cdot (-\hat{a}_x\, dy\, dz) + \int_0^1\int_0^1 \hat{a}_x(3+x)\Big|_{x=1} \cdot dy\, dz = -3+4 = 1$$

(b) $\underline{D} = \hat{a}_y(4+y^2)$

$$Q_e = \oiint_S \underline{D} \cdot d\underline{s} = \int_0^1\int_0^1 \hat{a}_y(4+y^2)\Big|_{y=0} \cdot (-\hat{a}_y\, dx\, dz) + \int_0^1\int_0^1 \hat{a}_y(4+y^2)\Big|_{y=1} \cdot \hat{a}_y\, dx\, dz = -4+5 = 1$$

1.4 $\underline{D}_2 = 6\hat{a}_x + 3\hat{a}_z$, $\chi_{e2} = \varepsilon_{sr2} - 1 = 2.56 - 1 = 1.56$

(a) $\underline{E}_2 = \frac{\underline{D}_2}{\varepsilon_{s2}} = \frac{1}{2.56\varepsilon_0}(6\hat{a}_x + 3\hat{a}_z) = \frac{1}{\varepsilon_0}\left(\frac{6}{2.56}\hat{a}_x + \frac{3}{2.56}\hat{a}_z\right)$

$\underline{E}_2 = \frac{1}{\varepsilon_0}(2.34\hat{a}_x + 1.1718\hat{a}_z)$

(b) $\underline{P}_2 = \varepsilon_0 \chi_e \underline{E}_2 = \varepsilon_0\left[1.56\frac{1}{\varepsilon_0}(2.34\hat{a}_x + 1.1718\hat{a}_z)\right]$

$\underline{P}_2 = 3.65\hat{a}_x + 1.828\hat{a}_z$

(c) $\underline{E}_{1x} = \underline{E}_{2x}$ Continuity of tangential components of $\underline{E}$-field

$\frac{D_{1x}}{\varepsilon_0} = \frac{D_{2x}}{\varepsilon_{r2}\varepsilon_0} \Rightarrow D_{1x} = \frac{D_{2x}}{\varepsilon_{r2}} = \frac{6}{2.56} = 2.344$

$\hat{n} \cdot (\underline{D}_2 - \underline{D}_1) = q_{es}$ Discontinuity of normal components of $\underline{D}$ density

$\hat{n} = \hat{a}_z$: $D_{2z} - D_{1z} = q_{es} = 0.2 \Rightarrow D_{1z} = D_{2z} - q_{es} = 3 - 0.2 = 2.8$

$D_{1z} = 2.8$

$\underline{D}_1 = 2.344\hat{a}_x + 2.8\hat{a}_z$

cont'd

$\boxed{1.4 \text{ cont'd}}$

(d) $\varepsilon_1 \underline{E}_1 = \underline{D}_1 \Rightarrow \varepsilon_0 \underline{E}_1 = \underline{D}_1 \Rightarrow \underline{E}_1 = \frac{1}{\varepsilon_0}\underline{D}_1 = \frac{1}{\varepsilon_0}(2.34\hat{a}_x + 2.8\hat{a}_z)$

$$\underline{E}_1 = \frac{1}{\varepsilon_0}(2.34\hat{a}_x + 2.8\hat{a}_z)$$

(e) $\chi_{e1} = \varepsilon_{sr1} - 1 = 1 - 1 = 0$

$$\underline{P}_1 = \varepsilon_0 \chi_{e1} \underline{E}_1 = 0$$

$\boxed{1.5}$ $\underline{H}_1 = 3\hat{a}_x + \hat{a}_z 9, \quad \mu_2 = 4\mu_0$

(a) $\underline{B}_1 = \mu_0 \underline{H}_1 = \mu_0(3\hat{a}_x + 9\hat{a}_z)$

(b) $\underline{M}_1 = \chi_{m1}\underline{H}_1, \quad \chi_{m1} = \mu_{sr} - 1 = 1 - 1 = 0$

$$\underline{M}_1 = 0$$

(c) $H_{2x} = H_{1x} = 3$ Continuity of tangential $\underline{H}$-field.

Continuity of normal $\underline{B}$ density

$$B_{2z} = B_{1z} = 9\mu_0 \Rightarrow \mu_2 H_{2z} = 9\mu_0 \Rightarrow 4\mu_0 H_{2z} = 9\mu_0$$

$$H_{2z} = \frac{9}{4} = 2.25$$

$$\underline{H}_2 = 3\hat{a}_x + 2.25\hat{a}_z$$

(d) $\underline{B}_2 = \mu_2 \underline{H}_2 = 4\mu_0(3\hat{a}_x + \frac{9}{4}\hat{a}_z) = \mu_0(12\hat{a}_x + 9\hat{a}_z)$

(e) $\underline{M}_2 = \chi_{m2}\underline{H}_2 \qquad \chi_{m2} = \mu_{2r} - 1 = 4 - 1 = 3$

$$\underline{M}_2 = 3(3\hat{a}_x + 2.25\hat{a}_z) = 9\hat{a}_x + 6.75\hat{a}_z$$

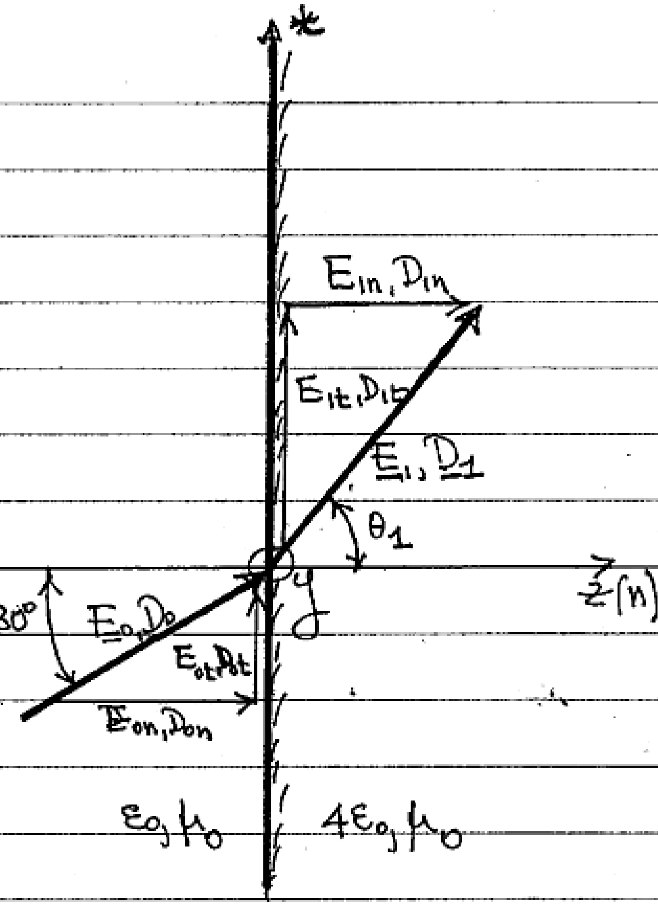
$\boxed{1.6}$ $D_o = \varepsilon_o E_o$

(a) $E_{on} = E_o \cos 30^o = 0.866 E_o$

$E_{ot} = E_o \sin 30^o = 0.5 E_o$

$D_{on} = \varepsilon_o E_{on} = 0.866 \varepsilon_o E_o$

$D_{ot} = \varepsilon_o E_{ot} = 0.5 \varepsilon_o E_o$



From B.Cs.

$E_{1t} = E_{ot} = 0.5 E_o$

$D_{1t} = \varepsilon_1 E_{1t} = 0.5(4)\varepsilon_o E_o$

$D_{1t} = 2\varepsilon_o E_o$

$D_{1n} = D_{on} = 0.866 \varepsilon_o E_o$

$E_{1n} = \frac{D_{1n}}{\varepsilon_1} = \frac{0.866 \varepsilon_o E_o}{4\varepsilon_o}$

$E_{1n} = 0.2165 E_o$

$$\boxed{E_1 = (0.5 \hat{a}_t + 0.2165 \hat{a}_n) E_o}$$

$$E_1 = \sqrt{(E_{1t})^2 + (E_{1n})^2} = \sqrt{(0.5)^2 + (0.2165)^2}\, E_o = \sqrt{0.25 + 0.046875}\, E_o$$

$$E_1 = \sqrt{0.296875}\, E_o = 0.54486 E_o$$

$$\boxed{E_1 = 0.54486 E_o}$$

$$\boxed{D_1 = (2\hat{a}_t + 0.866 \hat{a}_n)\varepsilon_o E_o}$$

$$D_1 = \sqrt{(D_{1t})^2 + (D_{1n})^2} = \sqrt{(2)^2 + (0.866)^2}\, \varepsilon_o E_o = \sqrt{4 + 0.749956}\, \varepsilon_o E_o$$

$$D_1 = \sqrt{4.749956}\, \varepsilon_o E_o = 2.17944\, \varepsilon_o E_o = 0.54486(4\varepsilon_o) E_o$$

$$\boxed{D_1 = 2.17944\, \varepsilon_o E_o = 0.54486(4\varepsilon_o) E_o}$$

(b) $\theta_1 = \tan^{-1}\left(\frac{E_{1t}}{E_{1n}}\right) = \tan^{-1}\left(\frac{0.5 E_o}{0.2165 E_o}\right) = \tan^{-1}(2.309) = \boxed{66.587^o}$

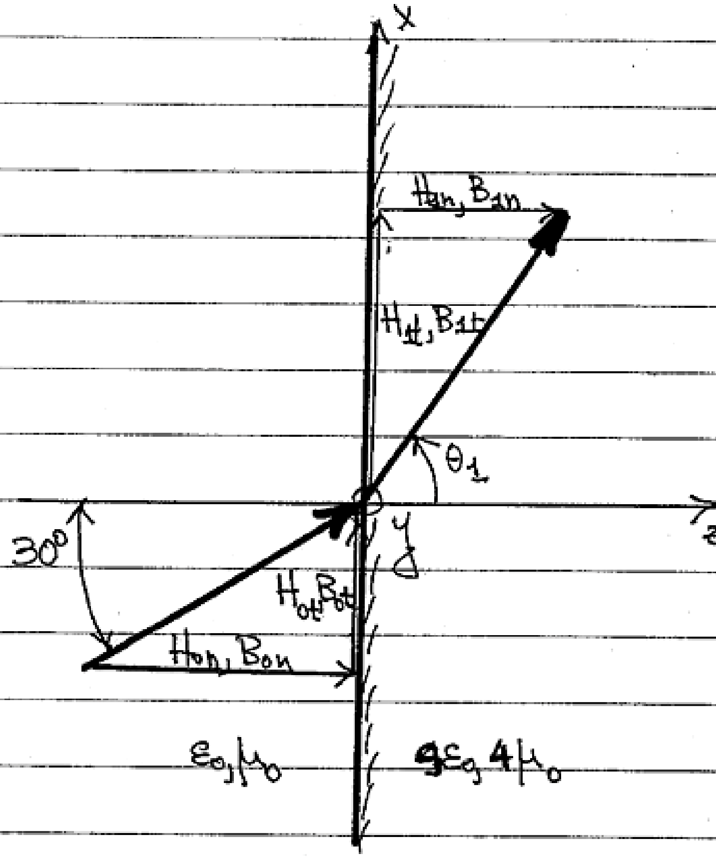
$\theta_1 = \tan^{-1}\left(\frac{D_{1t}}{D_{1n}}\right) = \tan^{-1}\left(\frac{2\varepsilon_o E_o}{0.866 \varepsilon_o E_o}\right) = \tan^{-1}\left(\frac{2}{0.86666}\right) = \tan^{-1}(2.308) = 66.571^o$

$$\boxed{\theta_1 = 66.571^o}$$

$\boxed{1.7}$ $B_o = \mu_o H_o$

$H_{on} = H_o \cos 30^\circ = 0.866 H_o$

$H_{ot} = H_o \sin 30^\circ = 0.5 H_o$



$B_{on} = \mu_o H_{on} = 0.866 \mu_o H_o$

$B_{ot} = \mu_o H_{ot} = 0.5 \mu_o H_o$

From B.Cs.

$H_{1t} = H_{ot} = 0.5 H_o$

$B_{1t} = \mu_1 H_{1t} = 0.5\, 4 \mu_o H_o$

$B_{1t} = 2 \mu_o H_o$

$B_{1n} = B_{on} = 0.866 \mu_o H_o$

$H_{1n} = \dfrac{B_{1n}}{\mu_1} = \dfrac{0.866 \mu_o H_o}{4 \mu_o}$

$H_{1n} = 0.2165 H_o$

$\boxed{\underline{H_1} = (0.5 \hat{a}_t + 0.2165 \hat{a}_n) H_o}$

$H_1 = \sqrt{(H_{1t})^2 + (H_{1n})^2} = \sqrt{(0.5)^2 + (0.2165)^2}\, H_o = \sqrt{0.296875}\, H_o$

$\boxed{H_1 = 0.54486 H_o}$

$\underline{B_1} = \boxed{(2 \hat{a}_t + 0.866 \hat{a}_n) \mu_o H_o}$

$B_1 = \sqrt{(B_{1t})^2 + (B_{1n})^2} = \sqrt{(2)^2 + (0.866)^2}\, \mu_o H_o = \sqrt{4.751}\, \mu_o H_o$

$\boxed{B_1 = 2.18 \mu_o H_o = 0.545 (4\mu_o) H_o}$

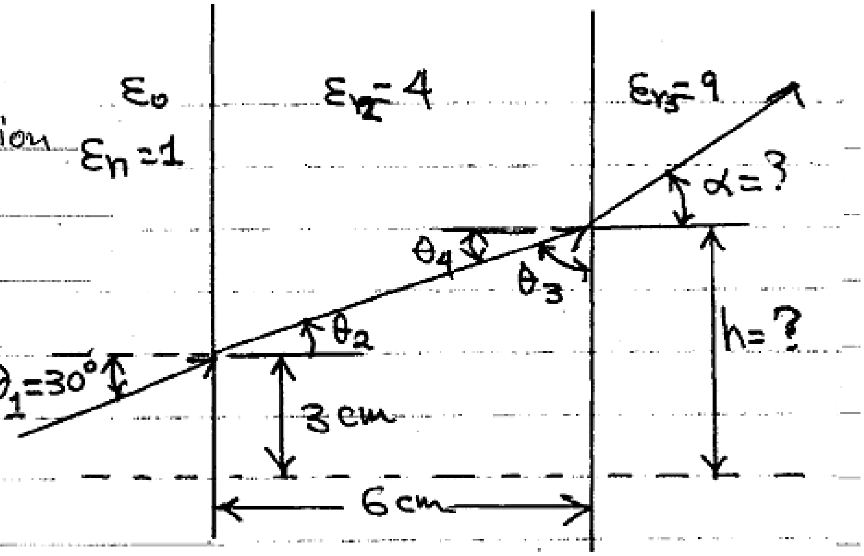
b. $\theta_1 = \tan^{-1}\left(\dfrac{H_{1t}}{H_{1n}}\right) = \tan^{-1}\left(\dfrac{0.5 H_o}{0.2165 H_o}\right) = \tan^{-1}(2.309) = 66.587^\circ$

$\boxed{\theta_1 = 66.587^\circ}$

$\theta_1 = \tan^{-1}\left(\dfrac{B_{1t}}{B_{1n}}\right) = \tan^{-1}\left(\dfrac{2 \mu_o H_o}{0.8666 \mu_o H_o}\right) = \tan^{-1}(2.308) = 66.571^\circ$

$\boxed{\theta_1 = 66.571^\circ}$

1.8 Snell's Law of Refraction



(a) $\beta_1 \sin\theta_1 = \beta_2 \sin\theta_2$

$\omega\sqrt{\mu_1\varepsilon_1}\sin\theta_1 = \omega\sqrt{\mu_2\varepsilon_2}\sin\theta_2$

Since $\mu_1 = \mu_2 = \mu_0$

$\sqrt{\varepsilon_1}\sin\theta_1 = \sqrt{\varepsilon_2}\sin\theta_2$

$\sin\theta_2 = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sin\theta_1$

$= \sqrt{\frac{1}{\varepsilon_2/\varepsilon_1}}\sin\theta_1$

$\sin\theta_2 = \sqrt{\frac{1}{4}}\sin\theta_1 = \frac{1}{2}\sin(30°) = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$

$\theta_2 = \sin^{-1}\left(\frac{1}{4}\right) = 14.4775° = \theta_4$

$\beta_2 \sin\theta_4 = \beta_3 \sin\alpha \Rightarrow \omega\sqrt{\mu_2\varepsilon_2}\sin\theta_4 = \omega\sqrt{\mu_3\varepsilon_3}\sin\alpha$

Since $\mu_2 = \mu_3 = \mu_0$:

$\sqrt{\varepsilon_3}\sin\alpha = \sqrt{\varepsilon_2}\sin\theta_4$

$\alpha = \sin^{-1}\left[\sqrt{\frac{\varepsilon_2}{\varepsilon_3}}\sin\theta_4\right] = \sin^{-1}\left(\sqrt{\frac{4}{9}}\sin\theta_4\right) = \sin^{-1}\left[\frac{2}{3}\sin(14.4775°)\right]$

$\alpha = \sin^{-1}\left[\frac{2}{3}(0.25)\right] = \sin^{-1}(0.1667)$

$$\boxed{\alpha = 9.594°}$$

(b) $\tan\theta_2 = \left(\frac{h-3}{6}\right) = \tan(14.4775°)$

$\frac{h-3}{6} = (0.2582) \Rightarrow h - 3 = 6(0.2582)$

$h = 3 + 6(0.2582) = 4.5492 \text{ cm}$

$$\boxed{h = 4.5492 \text{ cm}}$$

1.9 $\underline{D} = \varepsilon_0 \underline{E}$, $Q_e = \oiint_S \underline{D} \cdot d\underline{s}$

$$Q_e = \oiint_S \underline{D} \cdot d\underline{s} = \varepsilon_0 \iint_S \hat{a}_z E_z\Big|_{z=0} \cdot (-\hat{a}_z d\underline{s}) + \varepsilon_0 \iint_S \hat{a}_z E_z\Big|_{z=h} \cdot \hat{a}_z d\underline{s}$$

$$Q_e = -\varepsilon_0 \iint_S \left[-\frac{c}{h} - \frac{bh^2}{6\varepsilon_0}\right] ds + \varepsilon_0 \iint_S \left[-\frac{c}{h} + 2\frac{bh^2}{6\varepsilon_0}\right] ds = \frac{3h^2 b}{6}(\pi a^2) = \frac{\pi}{2} b (ha)^2$$

$\boxed{1.10}$ $\varepsilon_r = 4,\ \mu_r = 9,\ a = 4\text{ cm}$

$$\underline{H} = 3\hat{a}_\rho + 6\hat{a}_\phi + 8\hat{a}_z$$

(a) $\underline{B} = \mu\underline{H} = \mu_r\mu_o\underline{H} = 9\mu_o(3\hat{a}_\rho + 6\hat{a}_\phi + 8\hat{a}_z) = \mu_o(27\hat{a}_\rho + 54\hat{a}_\phi + 72\hat{a}_z)$

$$\underline{B} = = \mu_o(27\hat{a}_\rho + 54\hat{a}_\phi + 27\hat{a}_z)$$

(b) $\underline{H}_o = \hat{a}_\rho H_{\rho o} + \hat{a}_\phi H_{\phi o} + \hat{a}_z H_{zo}$

$$H_{\phi o} = H_\phi = 6$$

$$H_{zo} = H_z = 8$$

$$B_{\rho o} = B_\rho = \mu_o H_{\rho o} = \mu H_\rho = \mu_r\mu_o H_\rho$$

$$H_{\rho o} = \mu_r H_\rho = 9(3) = 27$$

$$\underline{H}_o = (27\hat{a}_\rho + 6\hat{a}_\phi + 8\hat{a}_z)$$

(c)

$$B_o = \mu_o\underline{H}_o = \mu_o(27\hat{a}_\rho + 6\hat{a}_\phi + 8\hat{a}_z)$$

$\boxed{1.11}$ $\nabla\cdot\underline{E} = 0$ for a source-free and homogeneous medium. Thus

$$\nabla\cdot\underline{E} = \left[\hat{a}_x\frac{\partial}{\partial x} + \hat{a}_y\frac{\partial}{\partial y} + \hat{a}_z\frac{\partial}{\partial z}\right]\cdot\left[\hat{a}_x A(x+y) + \hat{a}_y B(x-y)\right]\cos(\omega t) = 0$$

$$A\frac{\partial}{\partial x}(x+y) + B\frac{\partial}{\partial y}(x-y) = A(1) + B(-1) = 0 \Rightarrow A = B$$

Also

$$\underline{E} = \hat{a}_x A(x+y) + \hat{a}_y B(x-y)$$

$$\nabla\times\underline{E} = -j\omega\mu\underline{H} \Rightarrow \underline{H} = -\frac{1}{j\omega\mu}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A(x+y) & B(x-y) & 0 \end{vmatrix}$$

$$\underline{H} = -\frac{1}{j\omega\mu}\left[\hat{a}_x(0) + \hat{a}_y(0) + \hat{a}_z(B-A)\right]$$

$$\nabla\times\underline{H} = j\omega\varepsilon\underline{E} \Rightarrow \underline{E} = 0 \Rightarrow A = B = 0$$

$\boxed{1.12}$ $\quad \underline{B} = \hat{a}_z \dfrac{10^{-12}}{1+25\rho} \cos(1500\pi t)$

a. $$\Psi_m = \iint_{S_0} \underline{B} \cdot d\underline{s} = \int_0^{2\pi}\int_0^{a} \hat{a}_z B_z \cdot \hat{a}_z \rho \, d\rho \, d\phi = \int_0^{2\pi}\int_0^{a} B_z \rho \, d\rho \, d\phi$$

$$= 2\pi \int_0^{a} B_z \rho \, d\rho = 2\pi \times 10^{-12} \cos(1500\pi t) \int_0^{a} \frac{\rho}{1+25\rho} d\rho$$

Using the integral

$$\int \frac{u}{a+bu} du = \frac{1}{b^2}\left[a + bu - a \ln(a+bu)\right]$$

we can write the flux as

$$\Psi_m = 2\pi \times 10^{-12} \cos(1500\pi t) \left\{ \frac{1}{(25)^2} \left[1 + 25\rho - \ln(1+25\rho) \right]_0^a \right\}$$

$$= 2\pi \times 10^{-12} \cos(1500\pi t) \left\{ \frac{1}{625} \left[25a - \ln(1+25a) \right] \right\}$$

$$\Psi_m = 2\pi \times 10^{-12} \cos(1500\pi t) \left\{ \frac{1}{625} \left[2.5 - \ln(3.5) \right] \right\} = 1.2539 \times 10^{-14} \cos(1500\pi t)$$

b. $$\oint_C \underline{E} \cdot d\underline{l} = -\frac{\partial \Psi}{\partial t}$$

$$\int_0^{2\pi} (\hat{a}_\phi E_\phi) \cdot \hat{a}_\phi \rho \, d\phi = -\frac{\partial}{\partial t} \int_0^{2\pi}\int_0^{\rho} \underline{B} \cdot d\underline{s} = -\frac{\partial}{\partial t} \int_0^{2\pi}\int_0^{\rho} (\hat{a}_z B_z) \cdot \hat{a}_z \rho \, d\rho \, d\phi$$

$$2\pi \rho E_\phi = -2\pi \frac{\partial}{\partial t} \int_0^{\rho} B_z \rho \, d\rho = 2\pi \times 10^{-12} (1500\pi) \sin(1500\pi t) \int_0^{\rho} \frac{\rho}{1+25\rho} d\rho$$

$$\rho E_\phi = 1500\pi \times 10^{-12} \sin(1500\pi t) \left\{ \frac{1}{(25)^2} \left[1 + 25\rho - \ln(1+25\rho) \right]_0^{\rho} \right\}$$

$$E_\phi = \frac{7.5398 \times 10^{-12}}{\rho} \left[25\rho - \ln(1+25\rho) \right] \sin(1500\pi t)$$

<u>To check:</u> Use Maxwell's equation $\nabla \times \underline{E} = -\dfrac{\partial \underline{B}}{\partial t}$

$$\hat{a} \; \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho E_\phi) = -\frac{\partial}{\partial t} \left[\hat{a}_z \frac{10^{-12}}{1+25\rho} \cos(1500\pi t) \right]$$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left\{ 7.5398 \times 10^{-12} \left[25\rho - \ln(1+25\rho) \right] \sin(1500\pi t) \right\} = 1500\pi \times 10^{-12} \sin(1500\pi t) \frac{1}{1+25\rho}$$

$$\frac{4.712 \times 10^{-9}}{1+25\rho} \sin(1500\pi t) = \frac{4.712 \times 10^{-9}}{1+25\rho} \sin(1500\pi t)$$

QED

$\boxed{1.13}$ $\underline{\mathcal{B}} = \hat{a}_x B_x \cos(2y) \sin(\omega t - \pi z) + \hat{a}_y B_y \cos(2x) \cos(\omega t - \pi z)$

$$\nabla \times \underline{\mathcal{H}} = \cancel{\underline{\mathcal{J}}_i}^{\,0} + \cancel{\underline{\mathcal{J}}_c}^{\,0} + \underline{\mathcal{J}}_d = \underline{\mathcal{J}}_d = \frac{\partial \underline{\mathcal{D}}}{\partial t}$$

$$\underline{\mathcal{J}}_d = \nabla \times \underline{\mathcal{H}} = \frac{1}{\mu_0} \nabla \times \underline{\mathcal{B}} = \frac{1}{\mu_0} \left\{ -\hat{a}_x \frac{\partial \mathcal{B}_y}{\partial z} + \hat{a}_y \frac{\partial \mathcal{B}_x}{\partial z} + \hat{a}_z \left[\frac{\partial \mathcal{B}_y}{\partial x} - \frac{\partial \mathcal{B}_x}{\partial y} \right] \right\}$$

$$= \frac{1}{\mu_0} \left\{ -\hat{a}_x \pi B_y \cos(2x) \sin(\omega t - \pi z) - \hat{a}_y \pi B_x \cos(2y) \cos(\omega t - \pi z) + \hat{a}_z \left[-2B_y \sin(2x) \cos(\omega t - \pi z) + 2B_x \sin(2y) \sin(\omega t - \pi z) \right] \right\}$$

$$\underline{\mathcal{J}}_d = -\hat{a}_x 2.5 \times 10^6 B_y \cos(2x) \sin(\omega t - \pi z) - \hat{a}_y 2.5 \times 10^6 B_x \cos(2y) \cos(\omega t - \pi z) + \hat{a}_z \left[-1.59 \times 10^6 B_y \sin(2x) \cos(\omega t - \pi z) + 1.59 \times 10^6 B_x \sin(2y) \sin(\omega t - \pi z) \right]$$

$\boxed{1.14}$ $\underline{\mathcal{J}}_d = \hat{a}_x yz + \hat{a}_y y^2 + \hat{a}_z xyz$, $\quad \mathcal{I}_d = \oiint_S \underline{\mathcal{J}}_d \cdot d\underline{s}$

$$\mathcal{I}_d = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} (\hat{a}_x yz)\Big|_{x=-1/2} \cdot (-\hat{a}_x dy\,dz) + \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} (\hat{a}_x yz)\Big|_{x=1/2} \cdot (\hat{a}_x dy\,dz)$$
$$+ \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} (\hat{a}_y y^2)\Big|_{y=-1/2} \cdot (-\hat{a}_y dx\,dz) + \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} (\hat{a}_y y^2)\Big|_{y=1/2} \cdot (\hat{a}_y dx\,dz)$$
$$+ \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} (\hat{a}_z xyz)\Big|_{z=-1/2} \cdot (-\hat{a}_z dx\,dy) + \int_{-1/2}^{1/2} \int_{1/2}^{1/2} (\hat{a}_z xyz)\Big|_{z=1/2} \cdot (\hat{a}_z dx\,dy)$$

$$\mathcal{I}_d = 0 + 0 + 0 = 0$$

$\boxed{1.15}$ $\underline{\mathcal{D}} = \hat{a}_r \frac{10^{-9}}{4\pi} \frac{1}{r^2} \cos(\omega t - \beta r) = \mathrm{Re}\left[\hat{a}_r \frac{10^{-9}}{4\pi r^2} e^{j(\omega t - \beta r)} \right] = \mathrm{Re}\left[\underline{D} e^{j\omega t} \right]$

where $\underline{D} = \hat{a}_r \frac{10^{-9}}{4\pi r^2} e^{-j\beta r}$, $\quad \beta = \omega\sqrt{\mu_0 \varepsilon_0}$

$$Q_e = \oiint_S \underline{D} \cdot d\underline{s} = \int_0^{2\pi} \int_0^{\pi} (\hat{a}_r D_r) \cdot \hat{a}_r r^2 \sin\theta\, d\theta\, d\phi = \frac{10^{-9}}{4\pi} e^{-j\beta r} \int_0^{2\pi} \int_0^{\pi} \sin\theta\, d\theta\, d\phi$$

$$Q_e = \frac{10^{-9}}{4\pi} e^{-j\beta r}\, 2\pi \left(-\cos\theta \right)_0^{\pi} = 10^{-9} e^{-j\beta r}$$

$|Q_e| = 10^{-9}$ Coulombs

$\boxed{1.16}$ $\mathcal{E} = \mathrm{Re}\left[\underline{E}(r,\theta)e^{j\omega t}\right] = \mathrm{Re}\left[\hat{a}_\phi E_o \sin\theta \frac{e^{-j\beta_o r}}{r} e^{j\omega t}\right] = \hat{a}_\phi E_o \sin\theta \frac{\cos(\omega t - \beta_o r)}{r}$

where $\underline{E}(r,\theta) = \hat{a}_\phi E_o \sin\theta \frac{e^{-j\beta_o r}}{r}$

Using Maxwell's equation

$$\underline{H} = -\frac{1}{j\omega\mu_o}\nabla\times\underline{E} = -\frac{1}{j\omega\mu_o}\left[\hat{a}_r \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(E_\phi \sin\theta\right) - \hat{a}_\theta \frac{1}{r}\frac{\partial}{\partial r}\left(r E_\phi\right)\right]$$

$$= -\hat{a}_r \frac{2E_o}{j\omega\mu_o}\cos\theta \frac{e^{-j\beta_o r}}{r^2} - \hat{a}_\theta \sqrt{\frac{\varepsilon_o}{\mu_o}} E_o \sin\theta \frac{e^{-j\beta_o r}}{r}$$

$$\underline{H} \simeq -\hat{a}_\theta \sqrt{\frac{\varepsilon_o}{\mu_o}} E_o \sin\theta \frac{e^{-j\beta_o r}}{r} \Rightarrow \underline{\mathcal{H}} = \mathrm{Re}\left[\underline{H}e^{j\omega t}\right] \simeq -\hat{a}_\theta \sqrt{\frac{\varepsilon_o}{\mu_o}} E_o \sin\theta \frac{\cos(\omega t - \beta_o r)}{r}$$

$\boxed{1.17}$ $v(t) = 10\cos(\omega t)$, $\underline{\mathcal{J}}_c = \sigma\underline{\mathcal{E}}$, $\underline{\mathcal{J}}_d = \frac{\partial \mathcal{D}}{\partial t} = \varepsilon\frac{\partial \mathcal{E}}{\partial t}$

$\underline{\mathcal{E}}(t) = \frac{10}{2\times 10^{-2}}\cos(\omega t) = 500\cos(\omega t)$, $\frac{\partial \mathcal{E}}{\partial t} = -500\omega\sin(\omega t)$

a. $f = 1$ MHz

$$|\underline{\mathcal{J}}_c|_{max} = |\sigma\,\underline{\mathcal{E}}|_{max} = (3.7\times 10^{-4})500 = 0.185$$

$$|\underline{\mathcal{J}}_d|_{max} = \left|\varepsilon\frac{\partial\mathcal{E}}{\partial t}\right|_{max} = \left|-500\omega\varepsilon\sin(\omega t)\right|_{max} = \frac{2.56}{36} = 0.07111$$

b. $f = 100$ MHz

$$|\underline{\mathcal{J}}_c|_{max} = |\sigma\,\underline{\mathcal{E}}|_{max} = (3.7\times 10^{-4})500 = 0.185$$

$$|\underline{\mathcal{J}}_d|_{max} = \left|\varepsilon\frac{\partial\mathcal{E}}{\partial t}\right|_{max} = \frac{2.56(100)}{36} = 7.111$$

$\boxed{1.18}$ $\underline{\mathcal{E}} = \left[\hat{a}_y 5 + \hat{a}_z 10\right]\cos(\omega t - \beta x) = \mathrm{Re}\left[(\hat{a}_y 5 + \hat{a}_z 10)e^{j(\omega t - \beta x)}\right] = \mathrm{Re}\left[\underline{E}e^{j\omega t}\right]$

where $\underline{E} = (\hat{a}_y 5 + \hat{a}_z 10)e^{-j\beta x}$

a. $\underline{H} = -\frac{1}{j\omega\mu_o}\nabla\times\underline{E} = \hat{a}_y\left(-\frac{\partial E_z}{\partial x}\right) + \hat{a}_z\left(\frac{\partial E_y}{\partial x}\right) = 4.244\times 10^{-3}\left(-\hat{a}_y 10 + \hat{a}_z 5\right)e^{-j\beta x}$

$\underline{\mathcal{H}} = \mathrm{Re}\left[\underline{H}e^{j\omega t}\right] = 4.244\times 10^{-3}\left(-\hat{a}_y 10 + \hat{a}_z 5\right)\cos(\omega t - \beta x)$

(continued)

$\boxed{1.18 \text{ cont'd.}}$ The tangential components of the electric and magnetic fields must be continuous across the boundaries.

The normal components of the electric and magnetic fields must be discontinuous across the boundaries.

b. $\mathcal{E}_z^o(y=h^+) = 10\cos(\omega t - \beta x)$; $\mathcal{E}_y^o(y=h^+) = \frac{\varepsilon}{\varepsilon_o}(5)\cos(\omega t-\beta x) = 12.8\cos(\omega t - \beta x)$

$$\underline{\mathcal{E}}^o(y=h^+) = (\hat{a}_y 12.8 + \hat{a}_z 10)\cos(\omega t - \beta x)$$

$$\mathcal{H}_z^o(y=h^+) = 4.244\times10^{-3}(5)\cos(\omega t-\beta x);\ \mathcal{H}_y^o(y=h^+) = 4.244\times10^{-3}(-10)\cos(\omega t - \beta x)$$

$$\underline{\mathcal{H}}(y=h^+) = 4.244\times10^{-3}(-\hat{a}_y 10 + \hat{a}_z 5)\cos(\omega t - \beta x)$$

In a similar manner

$$\underline{\mathcal{E}}^o(y=-h^+) = (\hat{a}_y 12.8 + \hat{a}_z 10)\cos(\omega t - \beta x)$$

$$\underline{\mathcal{H}}(y=-h^-) = 4.244\times10^{-3}(-\hat{a}_y 10 + \hat{a}_z 5)\cos(\omega t - \beta x)$$

$\boxed{1.19}$ $\underline{\mathcal{J}} = \hat{a}_z 10^4 e^{-10^6 y}\cos(2\pi\times10^9 t)$

$$\underline{\mathcal{J}}(y=0.25\times10^{-3}) = \hat{a}_z 10^4 e^{-10^6(2.5\times10^{-4})}\cos(2\pi\times10^9 t) = \hat{a}_z 10^4 e^{-250}\cos(2\pi\times10^9 t) \simeq 0$$

$$I = \iint_S \underline{\mathcal{J}}\cdot d\underline{s} \simeq 2\int_0^{2.5\times10^{-4}}\int_0^{5\times10^{-3}} \left[\hat{a}_z 10^4 e^{-10^6 y}\cos(2\pi\times10^9 t)\right]\cdot\hat{a}_z\, dx\, dy$$

$$I \simeq 2(5\times10^{-3})(10^4)\cos(2\pi\times10^9 t)\int_0^{2.5\times10^{-4}} e^{-10^6 y}\,dy = 10^{-4}\cos(2\pi\times10^9 t)$$

$\boxed{1.20}$ a. $\underline{\mathcal{E}} = \hat{a}_y E_o \sin\left(\frac{\pi}{a}x\right)\cos(\omega t - \beta_z z) = \mathrm{Re}\left[\hat{a}_y E_o \sin\left(\frac{\pi}{a}x\right)e^{j(\omega t - \beta_z z)}\right] = \mathrm{Re}\left[\underline{E}e^{j\omega t}\right]$

where $\underline{E} = \hat{a}_y E_o \sin\left(\frac{\pi}{a}x\right)e^{-j\beta_z z}$

$$\underline{H} = -\frac{1}{j\omega\mu_o}\nabla\times\underline{E} = \hat{a}_x\frac{1}{j\omega\mu_o}\frac{\partial E_y}{\partial z} - \hat{a}_z\frac{1}{j\omega\mu_o}\frac{\partial E_y}{\partial x}$$

$$= -\hat{a}_x\frac{\beta_z}{\omega\mu_o}E_o\sin\left(\frac{\pi}{a}x\right)e^{-j\beta_z z} + \hat{a}_z j\frac{E_o}{\omega\mu_o}\left(\frac{\pi}{a}\right)\cos\left(\frac{\pi}{a}x\right)e^{-j\beta_z z}$$

$$\underline{\mathcal{H}} = \mathrm{Re}\left[\underline{H}e^{j\omega t}\right] = -\hat{a}_x\frac{\beta_z}{\omega\mu_o}E_o\sin\left(\frac{\pi}{a}x\right)\cos(\omega t - \beta_z z) + \hat{a}_z\frac{E_o}{\omega\mu_o}\left(\frac{\pi}{a}\right)\cos\left(\frac{\pi}{a}x\right)\cos\left(\omega t + \frac{\pi}{2} - \beta_z z\right)$$

b. Using $\nabla\times\underline{H} = j\omega\varepsilon_o\underline{E} \Rightarrow \hat{a}_y\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) = j\omega\varepsilon_o E_o\sin\left(\frac{\pi}{a}x\right)e^{-j\beta_z z}$

$$\hat{a}_y j\frac{E_o}{\omega\mu_o}\left[\beta_z^2 + \left(\frac{\pi}{a}\right)^2\right]\sin\left(\frac{\pi}{a}x\right)e^{-j\beta_z z} = j\omega\varepsilon_o E_o\sin\left(\frac{\pi}{a}x\right)e^{-j\beta_z z}$$

$$\omega\varepsilon_o = \frac{1}{\omega\mu_o}\left[\beta_z^2 + \left(\frac{\pi}{a}\right)^2\right] \Rightarrow \beta_z = \pm\sqrt{\omega^2\mu_o\varepsilon_o - \left(\frac{\pi}{a}\right)^2}$$

$\boxed{1.21}$ $\mathcal{E} = \hat{a}_\rho \left(\frac{100}{\rho}\right)\cos(10^8 t - \beta z) = \mathrm{Re}\left[\hat{a}_\rho \frac{100}{\rho} e^{j(10^8 t - \beta z)}\right] = \mathrm{Re}\left[\underline{E}\, e^{j\omega t}\right]$

where $\underline{E} = \hat{a}_\rho \frac{100}{\rho} e^{-j\beta z}$

a. $\underline{H} = -\frac{1}{j\omega\mu_0}\nabla\times\underline{E} = -\frac{1}{j\omega\mu_0}\left(\hat{a}_\phi \frac{\partial E_\rho}{\partial z}\right) = \hat{a}_\phi \frac{\beta}{\omega\mu}\frac{100}{\rho}e^{-j\beta z}$

$$\mathcal{H} = \mathrm{Re}\left[\underline{H}e^{j\omega t}\right] = \mathrm{Re}\left[\underline{H}e^{j10^8 t}\right] = \hat{a}_\phi \frac{\beta}{\omega\mu}\left(\frac{100}{\rho}\right)\cos(10^8 t - \beta z)$$

b. $\nabla\times\underline{H} = j\omega\varepsilon\,\underline{E} \Rightarrow -\hat{a}_\rho \frac{\partial H_\phi}{\partial z} = \hat{a}_\rho\, j\omega\varepsilon\, \frac{100}{\rho}e^{-j\beta z}$

$$\hat{a}_\rho\, j \frac{\beta^2}{\omega\mu_0}\left(\frac{100}{\rho}\right)e^{-j\beta z} = \hat{a}_\rho\, j\omega\varepsilon\left(\frac{100}{\rho}\right)e^{-j\beta z}$$

$$\omega\varepsilon = \frac{\beta^2}{\omega\mu_0} \Rightarrow \beta^2 = \omega^2\mu_0\varepsilon$$

c. $\mathcal{J}_d = \varepsilon\frac{\partial\mathcal{E}}{\partial t} = \varepsilon\frac{\partial}{\partial t}\left[\hat{a}_\rho \frac{100}{\rho}\cos(10^8 t - \beta z)\right] = -\hat{a}_\rho \frac{10^{10}\varepsilon}{\rho}\sin(10^8 t - \beta z)$

$$= -\hat{a}_\rho\, \varepsilon_r \frac{8.854\times10^{-2}}{\rho}\sin(10^8 t - \beta z) = -\hat{a}_\rho \frac{2.25(8.854\times10^{-2})}{\rho}\sin(10^8 t - \beta z)$$

$$\mathcal{J}_d = -\hat{a}_\rho \frac{0.1992}{\rho}\sin(10^8 t - \beta z)$$

$\boxed{1.22}$ $\mathcal{H} = \hat{a}_\phi \frac{2}{\rho}\cos\left(\frac{\pi}{2}z\right)\cos(4\pi\times10^8 t) = \mathrm{Re}\left[\hat{a}_\phi\frac{2}{\rho}\cos\left(\frac{\pi}{2}z\right)e^{j4\pi\times10^8 t}\right] = \mathrm{Re}\left[\underline{H}e^{j4\pi\times10^8 t}\right]$

where $\underline{H} = \hat{a}_\phi\frac{2}{\rho}\cos\left(\frac{\pi}{2}z\right)$, $\omega = 4\pi\times10^8$

a. $\underline{E} = \frac{1}{j\omega\varepsilon_0}\nabla\times\underline{H} = -\hat{a}_\rho\frac{1}{j\omega\varepsilon_0}\frac{\partial H_\phi}{\partial z} = -\hat{a}_\rho\, j\frac{1}{\omega\varepsilon_0}\left(\frac{\pi}{2}\right)\frac{2}{\rho}\sin\left(\frac{\pi}{2}z\right)$

$$\mathcal{E} = \mathrm{Re}\left[\underline{E}e^{j\omega t}\right] = \hat{a}_\rho \frac{1}{\omega\varepsilon_0}\left(\frac{\pi}{2}\right)\frac{2}{\rho}\sin\left(\frac{\pi}{2}z\right)\sin(4\pi\times10^8 t)$$

b. $\mathcal{J}_s = \hat{n}\times(\mathcal{H}_2 - \mathcal{H}_1)$: At $\rho=a$: $\mathcal{J}_s = \hat{a}_\rho\times(\hat{a}_\phi\mathcal{H}_\phi)\Big|_{\rho=a} = \hat{a}_z\frac{2}{a}\cos\left(\frac{\pi}{2}z\right)\cos(4\pi\times10^8 t)$

At $\rho=b$: $\mathcal{J}_s = -\hat{a}_\rho\times(\hat{a}_\phi\mathcal{H}_\phi)\Big|_{\rho=b} = -\hat{a}_z\frac{2}{b}\cos\left(\frac{\pi}{2}z\right)\cos(4\pi\times10^8 t)$

c. $\mathcal{J}_d = \varepsilon_0\frac{\partial\mathcal{E}_0}{\partial t} = \hat{a}_\rho\frac{4\pi\times10^8}{\omega}\left(\frac{\pi}{2}\right)\frac{2}{\rho}\sin\left(\frac{\pi}{2}z\right)\cos(4\pi\times10^8 t) = \hat{a}_\rho\frac{\pi}{2}\left(\frac{2}{\rho}\right)\sin\left(\frac{\pi}{2}z\right)\cos(4\pi\times10^8 t)$

d. $I_d = \int_0^\ell\int_0^{2\pi}\mathcal{J}_d\cdot d\underline{s} = \int_0^\ell\int_0^{2\pi}(\hat{a}_\rho\mathcal{J}_d)\cdot\hat{a}_\rho\,\rho\,d\phi\,dz = \frac{\pi}{2}(4\pi)\left[-\frac{2}{\pi}\cos\left(\frac{\pi}{2}z\right)\right]_0^\ell\cos(4\pi\times10^8 t)$

$$I_d = 8\pi\cos(4\pi\times10^8 t)$$

$\boxed{1.23}$ $\nabla \times \underline{\mathcal{E}} = -\underline{\mathcal{M}}_i - \frac{\partial \underline{\mathcal{B}}}{\partial t}$

Defining
$$\underline{\mathcal{E}} = \mathrm{Re}[\underline{E} e^{j\omega t}]$$
$$\underline{\mathcal{M}}_i = \mathrm{Re}[\underline{M}_i e^{j\omega t}]$$
$$\underline{\mathcal{B}} = \mathrm{Re}[\underline{B} e^{j\omega t}]$$

Thus
$$\nabla \times \left(\mathrm{Re}[\underline{E} e^{j\omega t}]\right) = -\mathrm{Re}[\underline{M}_i e^{j\omega t}] - \frac{\partial}{\partial t}\left(\mathrm{Re}[\underline{B} e^{j\omega t}]\right)$$

Interchanging differentiation with the Real part leads to
$$\mathrm{Re}\left[\nabla \times (\underline{E} e^{j\omega t})\right] = \mathrm{Re}\left[-\underline{M}_i e^{j\omega t}\right] + \mathrm{Re}\left[-\frac{\partial}{\partial t}(\underline{B} e^{j\omega t})\right]$$
or
$$\mathrm{Re}\left[(\nabla \times \underline{E}) e^{j\omega t}\right] = \mathrm{Re}\left[-\underline{M}_i e^{j\omega t}\right] + \mathrm{Re}\left[-j\omega \underline{B} e^{j\omega t}\right]$$

Lemma: If $\underline{A}$ and $\underline{B}$ are complex quantities and
$$\mathrm{Re}[\underline{A} e^{j\omega t}] = \mathrm{Re}[\underline{B} e^{j\omega t}] \quad \text{for all } t$$
then
$$\underline{A} = \underline{B}$$

Using this lemma, we can write that
$$\nabla \times \underline{E} = -\underline{M}_i - j\omega \underline{B}$$

The same procedure can be used for all the other differential form equations.

For the integral form
$$\oint_C \underline{\mathcal{E}} \cdot d\underline{\ell} = -\iint_S \underline{\mathcal{M}}_i \cdot d\underline{s} - \frac{\partial}{\partial t} \iint_S \underline{\mathcal{B}} \cdot d\underline{s}$$

Using the above definitions, we can write the integral form as
$$\oint_C \mathrm{Re}[\underline{E} e^{j\omega t}] \cdot d\underline{\ell} = -\iint_S \mathrm{Re}[\underline{M}_i e^{j\omega t}] \cdot d\underline{s} - \frac{\partial}{\partial t} \iint_S \mathrm{Re}[\underline{B} e^{j\omega t}] \cdot d\underline{s}$$
$$\mathrm{Re}\left\{\left[\oint \underline{E} \cdot d\underline{\ell}\right] e^{j\omega t}\right\} = \mathrm{Re}\left\{\left[-\iint_S \underline{M}_i \cdot d\underline{s}\right] e^{j\omega t}\right\} + \mathrm{Re}\left\{\left[-j\omega \iint_S \underline{B} \cdot d\underline{s}\right] e^{j\omega t}\right\}$$

Using the above lemma leads to
$$\oint_C \underline{E} \cdot d\underline{\ell} = -\iint_S \underline{M}_i \cdot d\underline{s} - j\omega \iint_S \underline{B} \cdot d\underline{s}$$

The same procedure can be used for all the other integral form equations.

$\boxed{1.24}$ $\underline{\mathcal{E}} = \mathrm{Re}\left[\underline{E}\, e^{j\omega t}\right] = \mathrm{Re}\left[(\underline{E}_R + j\underline{E}_x)(\cos\omega t + j\sin\omega t)\right]$

$$= \mathrm{Re}\left[(\underline{E}_R \cos\omega t - \underline{E}_x \sin\omega t) + j(\underline{E}_R \sin\omega t + \underline{E}_x \cos\omega t)\right]$$

$$\underline{\mathcal{E}} = (\underline{E}_R \cos\omega t - \underline{E}_x \sin\omega t)$$

Similarly

$$\underline{\mathcal{E}} = \frac{1}{2}\left[\underline{E}\, e^{j\omega t} + (\underline{E}\, e^{j\omega t})^*\right] = \frac{1}{2}\left[\underline{E}\, e^{j\omega t} + \underline{E}^* e^{-j\omega t}\right]$$

$$= \frac{1}{2}\left[(\underline{E}_R + j\underline{E}_x)(\cos\omega t + j\sin\omega t) + (\underline{E}_R - j\underline{E}_x)(\cos\omega t - j\sin\omega t)\right]$$

$$\underline{\mathcal{E}} = \frac{1}{2}\left[2(\underline{E}_R \cos\omega t - \underline{E}_x \sin\omega t)\right] = (\underline{E}_R \cos\omega t - \underline{E}_x \sin\omega t)$$

$\boxed{1.25}$

(a) $\underline{\mathcal{E}}(z,t) = \hat{a}_x E_0 \sin\left[(\omega t - \beta_0 z) - \frac{\pi}{2}\right]$

$$= \hat{a}_x E_0 \left[\sin(\omega t - \beta_0 z)\underbrace{\cos\left(-\frac{\pi}{2}\right)}_{0} + \cos(\omega t - \beta_0 z)\underbrace{\sin\left(-\frac{\pi}{2}\right)}_{-1}\right]$$

$$= -\hat{a}_x E_0 \cos(\omega t - \beta_0 z) = -\hat{a}_x E_0\, \mathrm{Re}\left[e^{j(\omega t - \beta_0 z)}\right]$$

$$\underline{\mathcal{E}}(z,t) = -\hat{a}_x E_0\, \mathrm{Re}\left[e^{j\omega t} e^{-j\beta_0 z}\right] = \mathrm{Re}\left[\underbrace{-\hat{a}_x E_0 e^{-j\beta_0 z}}_{E_x(z)} e^{j\omega t}\right]$$

$$\underline{E}(z) = -\hat{a}_x E_0 e^{-j\beta_0 z}$$

(b) $\nabla \times \underline{E} = -j\omega\mu_0 \underline{H} \Rightarrow \underline{H} = -\frac{1}{j\omega\mu_0}\nabla\times\underline{E} = -\frac{1}{j\omega\mu_0}\nabla\times\left[-\hat{a}_x E_0 e^{-j\beta_0 z}\right]$

$$\underline{H} = -\frac{1}{j\omega\mu_0}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = -\frac{1}{j\omega\mu_0}\left[\hat{a}_x(0) + \hat{a}_y\left(\frac{\partial E_x}{\partial z}\right) + \hat{a}_z\left(-\underbrace{\frac{\partial E_x}{\partial y}}_{0}\right)\right]$$

$$\underline{H} = -\frac{1}{j\omega\mu_0}\left[\hat{a}_y\left(\frac{\partial E_x}{\partial z}\right)\right] = -\hat{a}_y \frac{1}{j\omega\mu_0}\frac{\partial}{\partial z}\left[-E_0 e^{-j\beta_0 z}\right] = -\hat{a}_y E_0 \frac{\beta_0}{\omega\mu_0} e^{-j\beta_0 z}$$

$$\underline{H} = -\hat{a}_y E_0 \frac{\omega\sqrt{\mu_0\varepsilon_0}}{\omega\mu_0} e^{-j\beta_0 z} = -\hat{a}_y E_0 \sqrt{\frac{\varepsilon_0}{\mu_0}} e^{-j\beta_0 z} = -\hat{a}_y \frac{E_0}{\eta_0} e^{-j\beta_0 z}, \quad \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

(c)

$$\underline{S}_{ave} = \frac{1}{2}\mathrm{Re}\left(\underline{E}\times\underline{H}^*\right) = \frac{1}{2}\mathrm{Re}\left[\left(-\hat{a}_x E_0 e^{-j\beta_0 z}\right)\times\left(-\hat{a}_y E_0\sqrt{\frac{\varepsilon_0}{\mu_0}}e^{-j\beta_0 z}\right)^*\right]$$

$$\underline{S}_{ave} = \hat{a}_z \frac{1}{2}|E_0|^2\sqrt{\frac{\varepsilon_0}{\mu_0}} = \hat{a}_z \frac{1}{2\eta_0}|E_0|^2$$

$\boxed{1.26}$

$$\underline{H} = \hat{a}_\phi H_0 \frac{e^{-j\beta_0\rho}}{\sqrt{\rho}}$$

$$\underline{E} = \frac{1}{j\omega\varepsilon_0}\nabla\times\underline{H} = \frac{1}{j\omega\varepsilon_0}\left[\hat{a}_z \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\phi)\right] = \frac{H_0}{j\omega\varepsilon_0}\frac{\hat{a}_z}{\rho}\frac{\partial}{\partial\rho}\left[\rho^{1/2}e^{-j\beta_0\rho}\right]$$

$$= \hat{a}_z \frac{H_0}{j\omega\varepsilon_0}\frac{1}{\rho}\left[\rho^{1/2}(-j\beta_0 e^{-j\beta_0\rho}) + \frac{1}{2}\frac{1}{\rho^{1/2}}e^{-j\beta_0\rho}\right]$$

$$= \hat{a}_z H_0\left[-\frac{\beta_0}{\omega\varepsilon_0}\frac{e^{-j\beta_0\rho}}{\sqrt{\rho}} + \frac{e^{-j\beta_0\rho}}{j2\omega\varepsilon_0(\rho)^{3/2}}\right] \xrightarrow{\rho\to\text{large}} -\hat{a}_z H_0 \frac{\beta_0}{\omega\varepsilon_0}\frac{e^{-j\beta_0\rho}}{\sqrt{\rho}}$$

$$\underline{E} = -\hat{a}_z H_0 \frac{\omega\sqrt{\mu_0\varepsilon_0}}{\omega\varepsilon_0}\frac{e^{-j\beta_0\rho}}{\sqrt{\rho}} = -\hat{a}_z H_0\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{e^{-j\beta_0\rho}}{\sqrt{\rho}}$$

$\boxed{1.27}$

$$\underline{E} = \hat{a}_r E_r(r,\theta) + \hat{a}_\theta E_\theta(r,\theta),\ E_r = E_0\frac{\cos\theta}{r^2}\left(1+\frac{1}{j\beta_0 r}\right)e^{-j\beta_0 r},\ E_\theta = jE_0\frac{\beta_0\sin\theta}{2r}\left[1+\frac{1}{j\beta_0 r}-\frac{1}{(\beta_0 r)^2}\right]e^{-j\beta_0 r}$$

$$\underline{H} = -\frac{1}{j\omega\mu_0}\nabla\times\underline{E} = -\frac{1}{j\omega\mu_0}\left\{\frac{\hat{a}_\phi}{r}\left[\frac{\partial}{\partial r}(rE_\theta) - \frac{\partial E_r}{\partial\theta}\right]\right\}$$

Expanding using the above electric field components leads to

$$\underline{H} = \hat{a}_\phi j\sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{\beta_0\sin\theta}{2r}\left(1+\frac{1}{j\beta_0 r}\right)e^{-j\beta_0 r}$$

or $\quad H_r = H_\theta = 0$

$$H_\phi = j\sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{\beta_0\sin\theta}{2r}\left(1+\frac{1}{j\beta_0 r}\right)e^{-j\beta_0 r}$$

$\boxed{1.28}$ $\underline{E} = \hat{a}_\phi E_o \frac{\sin\theta}{r}\left(1+\frac{1}{j\beta_o r}\right)e^{-j\beta_o r} = \hat{a}_\phi E_\phi(r,\theta)$

$$\underline{H} = -\frac{1}{j\omega\mu_o}\nabla\times\underline{E} = -\frac{1}{j\omega\mu_o}\left\{\frac{\hat{a}_r}{r\sin\theta}\frac{\partial}{\partial\theta}\left(E_\phi\sin\theta\right) + \frac{\hat{a}_\theta}{r}\left[-\frac{\partial}{\partial r}\left(rE_\phi\right)\right]\right\}$$

Using the above electric field component leads to

$\underline{H} = \hat{a}_r H_r + \hat{a}_\theta H_\theta$ where $H_r = j\frac{2E_o\cos\theta}{\omega\mu_o r^2}\left(1+\frac{1}{j\beta_o r}\right)e^{-j\beta_o r}$

$$H_\theta = -\frac{E_o\sin\theta}{\eta r}\left[1+\frac{1}{j\beta_o r}-\frac{1}{(\beta_o r)^2}\right]e^{-j\beta_o r}$$

$\boxed{1.29}$ $\underline{E} = \hat{a}_z(1+j)\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{b}y\right)$

Using Maxwell's equation of $\nabla\times\underline{E} = -j\omega\mu_o\underline{H}$ leads to

$$\underline{H} = -\frac{1}{j\omega\mu_o}\left[\hat{a}_x\frac{\partial E_z}{\partial y} - \hat{a}_y\frac{\partial E_z}{\partial x}\right] = -\frac{(1+j)}{j\omega\mu_o}\left[\hat{a}_x\left(\frac{\pi}{b}\right)\cos\left(\frac{\pi}{b}y\right)\sin\left(\frac{\pi}{a}x\right) - \hat{a}_y\left(\frac{\pi}{a}\right)\sin\left(\frac{\pi}{b}y\right)\cos\left(\frac{\pi}{a}x\right)\right]$$

Now using Maxwell's equation of $\nabla\times\underline{H} = \underline{J}_c + j\omega\varepsilon\,\underline{E} = \sigma\underline{E} + j\omega\varepsilon\,\underline{E}$ leads to

$$\nabla\times\underline{H} = -\hat{a}_z\frac{(1+j)}{j\omega\mu_o}\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]\sin\left(\frac{\pi}{b}y\right)\sin\left(\frac{\pi}{a}x\right) = \hat{a}_z(\sigma+j\omega\varepsilon)(1+j)\sin\left(\frac{\pi}{b}y\right)\sin\left(\frac{\pi}{a}x\right)$$

Equating both sides, we can write that

$$(\sigma+j\omega\varepsilon) = -\frac{1}{j\omega\mu_o}\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right] \Rightarrow \begin{array}{l}\sigma = 0\\ \varepsilon_r = \frac{\varepsilon}{\varepsilon_o} = \frac{1}{\omega^2\mu_o\varepsilon_o}\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]\end{array}$$

$\boxed{1.30}$ $\underline{E}^i = \hat{a}_x e^{-j\beta_o z} \Rightarrow \underline{H}^i = -\frac{1}{j\omega\mu_o}\nabla\times\underline{E}^i = -\hat{a}_y\frac{1}{j\omega\mu_o}\frac{\partial E_x^i}{\partial z} = \hat{a}_y\sqrt{\frac{\varepsilon_o}{\mu_o}}\,e^{-j\beta_o z}$

$\underline{E}^r = -\hat{a}_x e^{+j\beta_o z} \Rightarrow \underline{H}^r = -\hat{a}_y\frac{1}{j\omega\mu_o}\frac{\partial E_x^r}{\partial z} = \hat{a}_y\sqrt{\frac{\varepsilon_o}{\mu_o}}\,e^{+j\beta_o z}$

$$\underline{J}_s = \hat{n}\times(\underline{H}_2-\underline{H}_1) = \hat{n}\times\underline{H}_2 = -\hat{a}_z\times\hat{a}_y\left(H^i+H^r\right)\Big|_{z=0} = \hat{a}_x\left(H^i+H^r\right)\Big|_{z=0} = \hat{a}_x\sqrt{\frac{\varepsilon_o}{\mu_o}}\left(e^{-j\beta_o z}+e^{+j\beta_o z}\right)\Big|_{z=0}$$

$$= \hat{a}_x 2\sqrt{\frac{\varepsilon_o}{\mu_o}} = \hat{a}_x\frac{2}{377} = \hat{a}_x\, 5.3\times10^{-3}\ \text{A/m}$$

$\boxed{1.31}$ $\underline{\underline{E}}^i = \hat{a}_y E_0 e^{-j\beta_0(x\sin\theta_i + z\cos\theta_i)}$, $\underline{\underline{E}}^r = \hat{a}_y E_0 \Gamma_h e^{-j\beta_0(x\sin\theta_i - z\cos\theta_i)}$

Along the interface $(z=0)$ $\left(\underline{\underline{E}}^i + \underline{\underline{E}}^r\right)^{tan}_{z=0} = 0 = \hat{a}_y E_0 (1+\Gamma_h) e^{-j\beta_0 x\sin\theta_i}$

which is satisfied provided $(1+\Gamma_h)=0 \Rightarrow \Gamma_h = -1$

$\boxed{1.32}$ Using Maxwell's equation of $\nabla \times \underline{\underline{E}} = -j\omega\mu_0 \underline{\underline{H}}$, the magnetic

a. field components corresponding to the electric fields of Problem 1.31 can be written as

$$\underline{\underline{H}}^i = \frac{E_0}{\sqrt{\mu_0/\varepsilon_0}}\left(-\hat{a}_x\cos\theta_i + \hat{a}_z\sin\theta_i\right) e^{-j\beta_0(x\sin\theta_i + z\cos\theta_i)}$$

$$\underline{\underline{H}}^r = -\frac{E_0}{\sqrt{\mu_0/\varepsilon_0}}\left(\hat{a}_x\cos\theta_i + \hat{a}_z\sin\theta_i\right) e^{-j\beta_0(x\sin\theta_i - z\cos\theta_i)}$$

b.

$$\underline{\underline{J}}_s = \hat{n}\times\left(\underline{\underline{H}}_2 - \underline{\underline{H}}_1\right)_{z=0} = -\hat{a}_z \times \left(\underline{\underline{H}}^i + \underline{\underline{H}}^r\right)_{z=0} = -\hat{a}_z \times \left[\hat{a}_x(H^i + H^r) + \hat{a}_z(H^i + H^r)\right]_{z=0}$$

$$\underline{\underline{J}}_s = -\hat{a}_z \times \hat{a}_x (H^i + H^r) = \hat{a}_y \frac{2E_0}{\sqrt{\mu_0/\varepsilon_0}}\cos\theta_i \, e^{-j\beta_0 x\sin\theta_i}$$

$\boxed{1.33}$ $\underline{\underline{E}}^i = (\hat{a}_x\cos\theta_i - \hat{a}_z\sin\theta_i) E_0 e^{-j\beta_0(x\sin\theta_i + z\cos\theta_i)}$

$\underline{\underline{E}}^r = (\hat{a}_x\cos\theta_i + \hat{a}_z\sin\theta_i)\Gamma_e e^{-j\beta_0(x\sin\theta_i - z\cos\theta_i)}$

Along the interface $(z=0)$ $\left(\underline{\underline{E}}^i + \underline{\underline{E}}^r\right)^{tan}_{z=0} = 0 = \hat{a}_x \cos\theta_i E_0 (1+\Gamma_e) e^{-j\beta_0 x\sin\theta_i}$

which is satisfied provided $(1+\Gamma_e)=0 \Rightarrow \Gamma_e = -1$

$\boxed{1.34}$ Along the interface the normal components of the electric flux density must be continuous; that is

$$\hat{n}\cdot(\underline{\underline{D}}_2 - \underline{\underline{D}}_1) = \hat{n}\cdot\underline{\underline{D}}_2 = -\hat{n}\cdot\hat{a}_z\varepsilon_0 E_0 \sin\theta_i (1+\Gamma_e) e^{-j\beta_0 x\sin\theta_i} = 0$$

$$= -\hat{a}_z\cdot\hat{a}_z\varepsilon_0 E_0\sin\theta_i(1+\Gamma_e)e^{-j\beta_0 x\sin\theta_i} = 0$$

which is satisfied provided

$$(1+\Gamma_e) = 0 \Rightarrow \Gamma_e = -1$$

$\boxed{1.35}$ Given the electric fields of Problem 1.33, the corresponding

a. magnetic field components can be found using Maxwell's equation of

$$\nabla \times \underline{E} = -j\omega\mu_0 \underline{H} \Rightarrow \underline{H} = -\frac{1}{j\omega\mu_0}\nabla\times\underline{E} = -\frac{1}{j\omega\mu_0}\left[\hat{a}_x\frac{\partial E_z}{\partial y} + \hat{a}_y\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{a}_z\frac{\partial E_x}{\partial y}\right]$$

For the incident field

$$E_x^i = \cos\theta_i\, E_0\, e^{-j\beta_0(x\sin\theta_i + z\cos\theta_i)}$$

$$E_y^i = 0$$

$$E_z^i = -\sin\theta_i\, E_0\, e^{-j\beta_0(x\sin\theta_i + z\cos\theta_i)}$$

Using these

$$\frac{\partial E_x^i}{\partial y} = 0 \quad , \quad \frac{\partial E_z^i}{\partial y} = 0$$

However

$$\frac{\partial E_x^i}{\partial z} = -j\beta_0\cos^2\theta_i\, E_0\, e^{-j\beta_0(x\sin\theta_i + z\cos\theta_i)}$$

$$\frac{\partial E_z^i}{\partial x} = +j\beta_0\sin^2\theta_i\, E_0\, e^{-j\beta_0(x\sin\theta_i + z\cos\theta_i)}$$

Thus we can write the incident magnetic field as

$$\underline{H}^i = -\frac{E_0}{j\omega\mu_0}\hat{a}_y(-j\beta_0)\left(\cos^2\theta_i + \sin^2\theta_i\right)e^{-j\beta_0(x\sin\theta_i + z\cos\theta_i)}$$

$$= \hat{a}_y\frac{\beta_0}{\omega\mu_0}E_0\, e^{-j\beta_0(x\sin\theta_i + z\cos\theta_i)} = \hat{a}_y\sqrt{\frac{\varepsilon_0}{\mu_0}}\,E_0\, e^{-j\beta_0(x\sin\theta_i + z\cos\theta_i)}$$

Using the same procedure we can write the reflected magnetic field as

$$\underline{H}^r = -\hat{a}_y\sqrt{\frac{\varepsilon_0}{\mu_0}}\,E_0\,\Gamma_e\, e^{-j\beta_0(x\sin\theta_i - z\cos\theta_i)} = \hat{a}_y\sqrt{\frac{\varepsilon_0}{\mu_0}}\,E_0\, e^{-j\beta_0(x\sin\theta_i - z\cos\theta_i)}$$

b. $\underline{J}_s = \hat{n}\times(\underline{H}_2 - \underline{H}_1) = \hat{n}\times\underline{H}_2 = -\hat{a}_z\times\hat{a}_y(H^i + H^r)\Big|_{z=0} = \hat{a}_x 2E_0\sqrt{\frac{\varepsilon_0}{\mu_0}}\, e^{-j\beta_0 x\sin\theta_i}$

$\boxed{1.36}$ To determine the coefficients Γ_0 and T_0 we apply the boundary conditions along the interface at $z=0$. To do this we first find the corresponding magnetic field components. This is accomplished using Maxwell's equation of $\nabla\times\underline{E} = -j\omega\mu_0\underline{H}_0 \Rightarrow \underline{H}_0 = -\frac{1}{j\omega\mu_0}\nabla\times\underline{E}$. Doing

(continued)

$\boxed{\text{1.36 cont'd}}$ this for each component leads to

$$\underline{H}^i = \hat{a}_y\sqrt{\frac{\varepsilon_0}{\mu_0}}E_0 e^{-j\beta_0 z},\quad \underline{H}^r = -\hat{a}_y\sqrt{\frac{\varepsilon_0}{\mu_0}}E_0\Gamma_0 e^{+j\beta_0 z},\quad \underline{H}^t = \hat{a}_y\sqrt{\frac{\varepsilon}{\mu_0}}E_0 T_0 e^{-j\beta z}$$

Applying the boundary conditions on the continuity of the tangential electric and magnetic fields along the interface at $z=0$ leads to

$1+\Gamma_0 = T_0$ from continuity of the electric fields

$1-\Gamma_0 = \sqrt{\varepsilon_r}\,T_0 = \sqrt{81}\,T_0$ from continuity of the magnetic fields

Solving these two equations, we find that

$$T_0 = \frac{2}{1+\sqrt{\varepsilon_r}} = \frac{2}{1+\sqrt{81}} = \frac{2}{1+9} = \frac{1}{5} = 0.2$$

$$\Gamma_0 = T_0 - 1 = \frac{1}{5} - 1 = -\frac{4}{5} = -0.8$$

$\boxed{1.37}$ The boundary conditions require continuity of the tangential components of the electric and magnetic fields.

Electric Fields: $\left(\underline{E}^i + \underline{E}^r\right)^{tan}_{z=0} = \left(\underline{E}^t\right)^{tan}_{z=0}$

$$1 + \Gamma_h = T_h$$

Magnetic Fields; $\left(\underline{H}^i + \underline{H}^r\right)^{tan}_{z=0} = \left(\underline{H}^t\right)^{tan}_{z=0}$

$$\cos\theta_i\left(-1+\Gamma_h\right)\sqrt{\varepsilon_0} = -\sqrt{1-\frac{\varepsilon_0}{\varepsilon}\sin^2\theta_i}\,\sqrt{\varepsilon}\;T_h$$

Solving these two equations leads to

$$\Gamma_h = \frac{\cos\theta_i - \sqrt{\frac{\varepsilon}{\varepsilon_0}}\sqrt{1-\frac{\varepsilon_0}{\varepsilon}\sin^2\theta_i}}{\cos\theta_i + \sqrt{\frac{\varepsilon}{\varepsilon_0}}\sqrt{1-\frac{\varepsilon_0}{\varepsilon}\sin^2\theta_i}}$$

$$T_h = \frac{2\cos\theta_i}{\cos\theta_i + \sqrt{\frac{\varepsilon}{\varepsilon_0}}\sqrt{1-\frac{\varepsilon_0}{\varepsilon}\sin^2\theta_i}}$$

1.38 The boundary conditions require continuity of the normal components of the electric flux density and magnetic flux density.

Electric Flux Density: $(\underline{D}^i+\underline{D}^r)^{nor}_{z=0}=(\underline{D}^t)^{nor}_{z=0}$

Since there are no normal components, this boundary condition is automatically satisfied.

Magnetic Flux Density: $(\underline{B}^i+\underline{B}^r)^{nor}_{z=0}=(\underline{B}^t)^{nor}_{z=0}$

$$\sin\theta_i\,\mu_o\sqrt{\varepsilon_o}\,(1+\Gamma_h)=\sqrt{\frac{\varepsilon_o}{\varepsilon}}\,\sin\theta_i\,\mu_o\sqrt{\varepsilon}\,T_h$$

or

$$1+\Gamma_h=T_h$$

This is identical to one of the equations for the solution of Problem 1.37 However we do not have another equation from the normal components of the electric field. Therefore we can not solve for Γ_h and T_h using only the normal components of Problem 1.37.

1.39 The boundary conditions require continuity of the tangential components of the electric and magnetic fields.

Electric Fields: $(\underline{E}^i+\underline{E}^r)^{tan}_{z=0}=(\underline{E}^t)^{tan}_{z=0}$

$$\cos\theta_i\,(1+\Gamma_e)=\sqrt{1-\frac{\varepsilon_o}{\varepsilon}\sin^2\theta_i}\;T_e$$

Magnetic Fields: $(\underline{H}^i+\underline{H}^r)^{tan}_{z=0}=(\underline{H}^t)^{tan}_{z=0}$

$$\sqrt{\varepsilon_o}\,(1-\Gamma_e)=\sqrt{\varepsilon}\;T_e$$

Solving these two equations leads to

$$\Gamma_e=\frac{-\cos\theta_i+\sqrt{\frac{\varepsilon_o}{\varepsilon}}\sqrt{1-\frac{\varepsilon_o}{\varepsilon}\sin^2\theta_i}}{\cos\theta_i+\sqrt{\frac{\varepsilon_o}{\varepsilon}}\sqrt{1-\frac{\varepsilon_o}{\varepsilon}\sin^2\theta_i}}$$

$$T_e=\frac{2\sqrt{\frac{\varepsilon_o}{\varepsilon}}\cos\theta_i}{\cos\theta_i+\sqrt{\frac{\varepsilon_o}{\varepsilon}}\sqrt{1-\frac{\varepsilon_o}{\varepsilon}\sin^2\theta_i}}$$

1.40 The boundary conditions require continuity of the normal components of the electric flux density and magnetic flux density.

Electric Flux Density: $(\underline{D}^i+\underline{D}^r)^{nor}_{z=0}=(\underline{D}^t)^{nor}_{z=0}$

$$\sin\theta_i\,\varepsilon_o(-1+\Gamma_e)=-\sqrt{\frac{\varepsilon_o}{\varepsilon}}\,\varepsilon\,\sin\theta_i\,T_e$$

or

$$\sqrt{\varepsilon_o}\,(1-\Gamma_e)=\sqrt{\varepsilon}\,T_e$$: This equation is identical to one of the equations for the solution of Problem 1.37.

Magnetic Flux Density: $(\underline{B}^i+\underline{B}^r)^{nor}_{z=0}=(\underline{D}^t)^{nor}_{z=0}$

Since there are no normal components, this boundary condition is automatically satisfied. However we only have one equation and two unknowns; therefore we can not solve for Γ_e and T_e using only the normal components of Problem 1.37.

1.41 From Problem 1.16 and at large distances

$$\underline{\mathcal{E}}=\hat{a}_\phi E_o\sin\theta\,\frac{\cos(\omega t-\beta r)}{r},\quad \underline{\mathcal{E}}=\mathrm{Re}[\underline{E}e^{j\omega t}]\Rightarrow \underline{E}=E_o\sin\theta\,\frac{e^{-j\beta_o r}}{r}\hat{a}_\phi$$

$$\underline{\mathcal{H}}=-\hat{a}_\theta\sqrt{\frac{\varepsilon_o}{\mu_o}}E_o\sin\theta\,\frac{\cos(\omega t-\beta_o r)}{r},\quad \underline{\mathcal{H}}=\mathrm{Re}[\underline{H}e^{j\omega t}]\Rightarrow \underline{H}=-\hat{a}_\theta\sqrt{\frac{\varepsilon_o}{\mu_o}}E_o\sin\theta\,\frac{e^{-j\beta_o r}}{r}$$

a. $$\underline{S}_{av}=\underline{S}=\frac{1}{2}\mathrm{Re}(\underline{E}\times\underline{H}^*)=\frac{1}{2}\mathrm{Re}(\hat{a}_\phi E_\phi\times\hat{a}_\theta H_\theta^*)=\frac{1}{2}\mathrm{Re}(\hat{a}_r E_\phi H_\theta^*)$$

$$=\hat{a}_r\frac{|E_o|^2}{2}\sqrt{\frac{\varepsilon_o}{\mu_o}}\,\frac{\sin^2\theta}{r^2}$$

b. $$P_{av}=\oiint_{S_a}\underline{S}_{av}\cdot d\underline{s}=\oiint_{S_a}\underline{S}_{av}\cdot\hat{a}_r r^2\sin\theta\,d\theta\,d\phi=\frac{|E_o|^2}{2}\sqrt{\frac{\varepsilon_o}{\mu_o}}\int_0^{2\pi}\int_0^{\pi}\sin^3\theta\,d\theta\,d\phi=\frac{|E_o|^2}{2}\sqrt{\frac{\varepsilon_o}{\mu_o}}(2\pi)\left(\frac{4}{3}\right)$$

$$=|E_o|^2\frac{4\pi}{3}\sqrt{\frac{\varepsilon_o}{\mu_o}}$$

1.42 $$\underline{H}^{inc}=\frac{1}{377}(-\hat{a}_x\cos\theta_i+\hat{a}_z\sin\theta_i),\quad \underline{H}^{ref}=\frac{1}{377}(-\hat{a}_x\cos\theta_i-\hat{a}_z\sin\theta_i)$$

$$\underline{H}^{total}=\underline{H}^{inc}+\underline{H}^{ref}=-\hat{a}_x\frac{2}{377}\cos\theta_i=-\hat{a}_x 5.31\times10^{-3}\cos\theta_i$$

$$\underline{J}_s=\hat{n}\times\underline{H}^{total}=\hat{a}_y\times(-\hat{a}_x 5.31\times10^{-3}\cos\theta_i)=\hat{a}_z\,5.31\times10^{-3}\cos\theta_i$$

1.43 $$\underline{E}=\hat{a}_y E_o\sin\left(\frac{\pi}{a}x\right)e^{-j\beta_z z}$$

a. $$\underline{H}=-\frac{1}{j\omega\mu_o}\nabla\times\underline{E}=-\frac{1}{j\omega\mu_o}\left(-\hat{a}_x\frac{\partial E_y}{\partial z}+\hat{a}_z\frac{\partial E_y}{\partial x}\right)=-\hat{a}_x\frac{\beta_z}{\omega\mu_o}E_o\sin\left(\frac{\pi}{a}x\right)e^{-j\beta_z z}-\hat{a}_z\frac{E_o}{j\omega\mu_o}\left(\frac{\pi}{a}\right)\cos\left(\frac{\pi}{a}x\right)e^{-j\beta_z z}$$

b. $$P_s=-\frac{1}{2}\iiint_V(\underline{H}^*\cdot\cancelto{0}{\underline{M}_i}+\underline{E}\cdot\cancelto{0}{\underline{J}_i^*})\,dv=0$$

c. $$P_e=\oiint_S\left(\frac{1}{2}\underline{E}\times\underline{H}^*\right)\cdot d\underline{s}$$

Cont'd

$\boxed{\text{1.43 cont'd}}$ $\frac{1}{2}\underline{E}\times\underline{H}^* = \frac{1}{2}\hat{a}_y E_y \times\left(-\hat{a}_x H_x^* - \hat{a}_z H_z^*\right) = \frac{1}{2}\left(\hat{a}_z E_y H_x^* - \hat{a}_x E_y H_z^*\right)$

$$\underline{S} = \frac{1}{2}\underline{E}\times\underline{H}^* = \hat{a}_z \frac{\beta_z}{2\omega\mu_o}|E_o|^2\sin^2\left(\frac{\pi}{a}x\right) + \hat{a}_x \frac{|E_o|^2}{j2\omega\mu_o}\left(\frac{\pi}{a}\right)\sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{a}x\right)$$

<u>From the two side walls at x=0 and x=a:</u>

$$\int_0^1\int_0^b (\hat{a}_x S_x)\Big|_{x=0}\cdot(-\hat{a}_x\,dy\,dz) + \int_0^1\int_0^b (\hat{a}_x S_x)\Big|_{x=a}\cdot(\hat{a}_x\,dy\,dz) = 0+0=0$$

<u>From the front and back cross sections at z=0 and z=1:</u>

$$\int_0^b\int_0^a (\hat{a}_z S_z)\Big|_{z=0}\cdot(+\hat{a}_z\,dx\,dy) = \frac{\beta_z}{2\omega\mu_o}|E_o|^2\int_0^b\int_0^a \sin^2\left(\frac{\pi}{a}x\right)dx\,dy = \frac{\beta_z}{2\omega\mu_o}\frac{ab}{2}|E_o|^2$$

$$\int_0^b\int_0^a (\hat{a}_z S_z)\Big|_{z=1}\cdot(-\hat{a}_z\,dx\,dy) = -\frac{\beta_z}{2\omega\mu_o}|E_o|^2\int_0^b\int_0^a \sin^2\left(\frac{\pi}{a}x\right)dx\,dy = -\frac{\beta_z}{2\omega\mu_o}\frac{ab}{2}|E_o|^2$$

<u>From the top and bottom walls at y=0 and y=b:</u>

Since there are no y components of the power density, there is no contribution from the top and bottom walls.

Therefore

$$P_e = 0 + \frac{\beta_z}{2\omega\mu_o}\frac{ab}{2}|E_o|^2 - \frac{\beta_z}{2\omega\mu_o}\frac{ab}{2}|E_o|^2 = 0$$

d. $P_d = \frac{1}{2}\iiint_V \sigma|\underline{E}|^2 dv = 0$

e. $$\overline{W}_m = \iiint_V \frac{1}{4}\mu_o|\underline{H}|^2 dv = \frac{\mu_o}{4}|E_o|^2\left\{\int_0^1\int_0^b\int_0^a\left(\frac{\beta_z}{\omega\mu_o}\right)^2\sin^2\left(\frac{\pi}{a}x\right)dx\,dy\,dz + \left(\frac{1}{\omega\mu_o}\right)^2\left(\frac{\pi}{a}\right)^2\int_0^1\int_0^b\int_0^a\cos^2\left(\frac{\pi}{a}x\right)dx\,dy\,dz\right\}$$

$$= \frac{\mu_o}{4}|E_o|^2\left\{\left(\frac{\beta_z}{\omega\mu_o}\right)^2\frac{ab}{2} + \left(\frac{1}{\omega\mu_o}\right)^2\left(\frac{\pi}{a}\right)^2\left(\frac{ab}{2}\right)\right\} = \frac{\mu_o}{4}|E_o|^2\left\{\frac{\varepsilon_o}{\mu_o}\frac{ab}{2} - \left(\frac{1}{\omega\mu_o}\right)^2\left(\frac{\pi}{a}\right)^2\frac{ab}{2} + \left(\frac{1}{\omega\mu_o}\right)^2\left(\frac{\pi}{a}\right)^2\frac{ab}{2}\right\}$$

$$\overline{W}_m = \frac{\varepsilon_o}{8}ab|E_o|^2$$

f. $$\overline{W}_e = \iiint_V \frac{1}{4}\varepsilon_o|\underline{E}|^2 dv = \frac{\varepsilon_o}{4}|E_o|^2\int_0^1\int_0^b\int_0^a \sin^2\left(\frac{\pi}{a}x\right)dx\,dy\,dz = \frac{\varepsilon_o}{4}|E_o|^2\frac{ab}{2} = \frac{\varepsilon_o}{8}ab|E_o|^2$$

$$\overline{W}_e = \frac{\varepsilon_o}{8}ab|E_o|^2$$

Ultimately $P_s = P_e + P_d + j2\omega(\overline{W}_m - \overline{W}_e)$

$0 = 0 + 0 + j2\omega\left(\frac{\varepsilon_o}{8}ab|E_o|^2 - \frac{\varepsilon_o}{8}ab|E_o|^2\right) = 0+0+0 = 0$

$0 = 0$ QED

$\boxed{1.44}$ $$-\nabla\cdot\left(\tfrac{1}{2}\underline{E}\times\underline{H}^*\right)=\tfrac{1}{2}\underline{H}^*\cdot\cancel{\underline{M}_i}^{\,0}+\tfrac{1}{2}\underline{E}\cdot\cancel{\underline{J}_i^*}^{\,0}+j2\omega\left(\tfrac{1}{4}\mu_0|\underline{H}|^2-\tfrac{1}{4}\epsilon_0|\underline{E}|^2\right)$$

$$\tfrac{1}{2}(\underline{E}\times\underline{H}^*)=\hat{a}_x\frac{|E_0|^2}{j2\omega\mu_0}\left(\frac{\pi}{a}\right)\sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{a}x\right)+\hat{a}_z\frac{\beta_z}{2\omega\mu_0}|E_0|^2\sin^2\left(\frac{\pi}{a}x\right)$$

$$=\hat{a}_x\frac{|E_0|^2}{j4\omega\mu_0}\left(\frac{\pi}{a}\right)\sin\left(\frac{2\pi}{a}x\right)+\hat{a}_z\frac{\beta_z}{2\omega\mu_0}|E_0|^2\sin^2\left(\frac{\pi}{a}x\right)$$

$$-\nabla\cdot\left(\tfrac{1}{2}\underline{E}\times\underline{H}^*\right)=-\frac{|E_0|^2}{j4\omega\mu_0}\left(\frac{\pi}{a}\right)\left(\frac{2\pi}{a}\right)\cos\left(\frac{2\pi}{a}x\right)=j\frac{|E_0|^2}{2\omega\mu_0}\left(\frac{\pi}{a}\right)^2\cos\left(\frac{2\pi}{a}x\right)$$

$$\tfrac{1}{4}\mu_0|\underline{H}|^2=\frac{\mu_0}{4}\frac{|E_0|^2}{(\omega\mu_0)^2}\left[\beta_z^2\sin^2\left(\frac{\pi}{a}x\right)+\left(\frac{\pi}{a}\right)^2\cos^2\left(\frac{\pi}{a}x\right)\right]$$

$$=\frac{\mu_0}{4}\frac{|E_0|^2}{(\omega\mu_0)^2}\left[\beta_0^2\sin^2\left(\frac{\pi}{a}x\right)+\left(\frac{\pi}{a}\right)^2\left(\cos^2\frac{\pi}{a}x-\sin^2\frac{\pi}{a}x\right)\right]$$

$$=\frac{\mu_0}{4}\frac{|E_0|^2}{(\omega\mu_0)^2}\left[\beta_0^2\sin^2\left(\frac{\pi}{a}x\right)+\left(\frac{\pi}{a}\right)^2\left(\frac{1+\cos\left(\frac{2\pi}{a}x\right)}{2}-\frac{1-\cos\left(\frac{2\pi}{a}x\right)}{2}\right)\right]$$

$$\tfrac{1}{4}\mu_0|\underline{H}|^2=\frac{|E_0|^2}{4}\epsilon_0\sin^2\left(\frac{\pi}{a}x\right)+\frac{|E_0|^2}{4}\frac{1}{\omega^2\mu_0}\left(\frac{\pi}{a}\right)^2\cos\left(\frac{2\pi}{a}x\right)$$

$$\tfrac{1}{4}\epsilon_0|\underline{E}|^2=\frac{|E_0|^2}{4}\epsilon_0\sin^2\left(\frac{\pi}{a}x\right)$$

Therefore the conservation of energy equation in differential form can be written as

$$j\frac{|E_0|^2}{2\omega\mu_0}\left(\frac{\pi}{a}\right)^2\cos\left(\frac{2\pi}{a}x\right)=0+0+j2\omega\left[\frac{|E_0|^2}{4}\epsilon_0\sin^2\left(\frac{\pi}{a}x\right)+\frac{|E_0|^2}{4}\frac{1}{\omega^2\mu_0}\left(\frac{\pi}{a}\right)^2\cos\left(\frac{2\pi}{a}x\right)-\frac{|E_0|^2}{4}\epsilon_0\sin^2\left(\frac{\pi}{a}x\right)\right]$$

$$j\frac{|E_0|^2}{2\omega\mu_0}\left(\frac{\pi}{a}\right)^2\cos\left(\frac{2\pi}{a}x\right)=j\frac{|E_0|^2}{2\omega\mu_0}\left(\frac{\pi}{a}\right)^2\cos\left(\frac{2\pi}{a}x\right)\qquad \text{QED}$$

$\boxed{1.45}$ Boundary Conditions on PEC:

(a) $E_x(0\le x\le a,\ y=0,\ 0\le z\le c)=E_x(\text{bottom wall})=0$

$$E_x=\cos(\beta_x x)\sin(\beta_y y)\sin(\beta_z z)\Big|_{y=0}=0$$

$$E_x(0\le x\le a,\ y=b,\ 0\le z\le c)=E_x(\text{top wall})=0$$

$$E_x=\cos(\beta_x x)\sin(\beta_y b)\sin(\beta_z z)=0$$

$$\sin(\beta_y b)=0\Rightarrow\beta_y b=\sin^{-1}(0)=n\pi,\quad n=0,\pm1,\pm2,\ldots$$

$$\boxed{\beta_y=(n\pi/b),\quad n=0,\pm1,\pm2,\ldots}$$

$\boxed{\text{Cont'd}}$

(b) $E_y(x=0,\ 0\le y\le b,\ 0\le z\le c) = E_y(\text{left wall}) = 0$

$$E_y = \sin(0)\cos(\beta_y y)\sin(\beta_z z) = 0$$

$E_y(x=a,\ 0\le y\le b,\ 0\le z\le c) = E_y(\text{right wall}) = 0$

$$E_y = \sin(\beta_x a)\cos(\beta_y y)\sin(\beta_z z) = 0$$

$$\sin(\beta_x a) = 0 \Rightarrow \beta_x a = \sin^{-1}(0) = m\pi,\ m = 0, \pm1, \pm2, \ldots$$

$$\boxed{\beta_x = (m\pi/a),\ m = 0, \pm1, \pm2, \ldots}$$

($m = n \ne 0$ simultaneously; $m = n = 0$ trivial solution, E-fields vanish)

c. $E_y(0\le x\le a,\ 0\le y\le b,\ z=0) = E_y(\text{front wall}) = 0$

$$E_y = \sin(\beta_x x)\cos(\beta_y y)\sin(0) = 0$$

$E_y(0\le x\le a,\ 0\le y\le b,\ z=c) = E_y(\text{back wall}) = 0$

$$E_y = \sin\beta_x x\cos(\beta_y y)\sin(\beta_z c) = 0$$

$$\sin(\beta_z c) = 0 \Rightarrow \beta_z c = \sin^{-1}(0) = p\pi,\ p = \cancel{0}, \pm1, \pm2, \ldots$$

$$\boxed{\beta_z = (p\pi/c) = (p\pi/c),\quad p = \pm1, \pm2, \ldots}$$

$p \ne 0$, because $p = 0$ leads to trivial solution; E-fields vanish

1.46 $\underline{E} = \hat{a}_y E_o \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{c}z\right)$, $\omega_r = \omega = \frac{1}{\sqrt{\mu_o\epsilon_o}}\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{c}\right)^2}$

a. $\underline{H} = -\frac{1}{j\omega\mu_o}\nabla\times\underline{E} = -\frac{1}{j\omega\mu_o}\left[-\hat{a}_x\frac{\partial E_y}{\partial z} + \hat{a}_z\frac{\partial E_y}{\partial x}\right]$

$$= \hat{a}_x \frac{E_o}{j\omega\mu_o}\left(\frac{\pi}{c}\right)\sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{c}z\right) - \hat{a}_z\frac{E_o}{j\omega\mu_o}\left(\frac{\pi}{a}\right)\cos\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{c}z\right)$$

b. $P_s = -\frac{1}{2}\iiint_V \left[\underline{H}^*\cdot\underline{M}_i + \underline{E}\cdot\underline{J}_i^*\right] dv = 0$

(continued)

$\boxed{\text{1.46 cont'd.}}$

c. $P_e = \oiint_{S_a} (\frac{1}{2}\underline{E}\times\underline{H}^*)\cdot d\underline{s} = \oiint_{S_a} \underline{S}\cdot d\underline{s}$, $\underline{S} = \frac{1}{2}\underline{E}\times\underline{H}^*$

$$\underline{S} = \frac{1}{2}\underline{E}\times\underline{H}^* = \frac{1}{2}\hat{a}_y E_y \times (\hat{a}_x H_x^* - \hat{a}_z H_z^*) = \frac{1}{2}(-\hat{a}_z E_y H_x^* - \hat{a}_x E_y H_z^*)$$

$$= \frac{1}{2}\left[\hat{a}_x \frac{E_0}{j\omega\mu_0}\left(\frac{\pi}{a}\right)\sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{a}x\right)\sin^2\left(\frac{\pi}{c}z\right) + \hat{a}_z \frac{|E_0|^2}{j\omega\mu_0}\left(\frac{\pi}{c}\right)\sin^2\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{c}z\right)\cos\left(\frac{\pi}{c}z\right)\right]$$

<u>Contributions to P_e from the different walls:</u>

<u>Left and right walls:</u>

$$\int_0^c\int_0^b (\hat{a}_x S_x)\Big|_{x=0}\cdot(-\hat{a}_x\,dy\,dz) = 0\ ; \quad \int_0^c\int_0^b (\hat{a}_x S_x)\Big|_{x=0}\cdot(\hat{a}_x\,dy\,dz) = 0$$

<u>Back and front walls:</u>

$$\int_0^b\int_0^a (\hat{a}_z S_z)\Big|_{z=0}\cdot(-\hat{a}_z\,dx\,dy) = 0\ ; \quad \int_0^b\int_0^a (\hat{a}_z S_z)\Big|_{z=0}\cdot(\hat{a}_z\,dx\,dy) = 0$$

<u>Top and bottom walls:</u>

Since there are no y components of the power density, there are no contributions from the top and bottom walls.

Therefore $P_e = 0+0+0+0+0 = 0$

d. $P_d = \frac{1}{2}\iiint_V \sigma|\underline{E}|^2 dv = 0$

e. $$\bar{W}_m = \frac{\mu_0}{4}\iiint_V |\underline{H}|^2 dv = |E_0|^2\frac{\mu_0}{4}\frac{1}{(\omega\mu_0)^2}\left[\left(\frac{\pi}{c}\right)^2\int_0^c\int_0^b\int_0^a \sin^2\left(\frac{\pi}{a}x\right)\cos^2\left(\frac{\pi}{c}z\right)dx\,dy\,dz + \left(\frac{\pi}{a}\right)^2\int_0^c\int_0^b\int_0^a \cos^2\left(\frac{\pi}{a}x\right)\sin^2\left(\frac{\pi}{c}z\right)dx\,dy\,dz\right]$$

$$\bar{W}_m = |E_0|^2\frac{\mu_0}{4}\frac{1}{(\omega\mu_0)^2}\left[\left(\frac{\pi}{c}\right)^2\frac{abc}{4} + \left(\frac{\pi}{a}\right)^2\frac{abc}{4}\right] = |E_0|^2\frac{abc}{16}\frac{\varepsilon_0}{\omega^2\mu_0\varepsilon_0}\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{c}\right)^2\right] = |E_0|^2\frac{abc}{16}\varepsilon_0$$

f. $$\bar{W}_e = \frac{\varepsilon_0}{4}\iiint_V |\underline{E}|^2 dv = |E_0|^2\frac{\varepsilon_0}{4}\int_0^c\int_0^b\int_0^a \sin^2\left(\frac{\pi}{a}x\right)\sin^2\left(\frac{\pi}{c}z\right)dx\,dy\,dz = |E_0|^2\frac{\varepsilon_0}{4}\,\frac{abc}{4} = |E_0|^2\frac{abc}{16}\varepsilon_0$$

Ultimately
$$P_s = P_e + P_d + j2\omega(\bar{W}_m - \bar{W}_e)$$
$$0 = 0 + 0 + j2\omega\left(|E_0|^2\frac{abc}{16}\varepsilon_0 - |E_0|^2\frac{abc}{16}\varepsilon_0\right)$$
$$0 = 0$$

QED