Part A. ORDINARY DIFFERENTIAL EQUATIONS (ODEs)

CHAPTER 1 First-Order ODEs

Major Changes

There is more material on modeling in the text as well as in the problem set.

Some additions on population dynamics appear in Sec. 1.5.

Team Projects, CAS Projects, and CAS Experiments are included in most problem sets.

SECTION 1.1. Basic Concepts. Modeling, page 2

Purpose. To give the students a first impression of what an ODE is and what we mean by solving it.

The role of initial conditions should be emphasized since, in most cases, solving an engineering problem of a physical nature usually means finding the solution of an initial value problem (IVP).

Further points to stress and illustrate by examples are:

The fact that a general solution represents a family of curves.

The distinction between an arbitrary constant, which in this chapter will always be denoted by c, and a fixed constant (usually of a physical or geometric nature and given in most cases).

The examples of the text illustrate the following.

Example 1: the verification of a solution

Examples 2 and 3: ODEs that can actually be solved by calculus with Example 2 giving an impression of exponential growth (Malthus!) and decay (radioactivity and further applications in later sections)

Example 4: the straightforward solution of an IVP

Example 5: a very detailed solution in all steps of a physical IVP involving a physical constant k

Background Material. For the whole chapter we need integration formulas and techniques from calculus, which the student should review.

General Comments on Text

This section should be covered relatively rapidly to get quickly to the actual solution methods in the next sections.

Equations (1)–(3) are just examples, not for solution, but the student will see that solutions of (1) and (2) can be found by calculus. Instead of (3), one could perhaps take a third-order linear ODE with constant coefficients or an Euler–Cauchy equation, both not of great interest. The present (3) is included to have a nonlinear ODE (a concept that will be mentioned later when we actually need it); it is not too difficult to verify that a solution is

$$y = \frac{ax + b}{cx + d}$$

with arbitrary constants a, b, c, d.

Problem Set 1.1 will help the student with the tasks of

Solving $y' = f(x)$ by calculus

Finding particular solutions from given general solutions

Setting up an ODE for a given function as solution, e.g., $y = e^x$

Gaining a first experience in modeling, by doing one or two problems

Gaining a first impression of the importance of ODEs without wasting time on matters that can be done much faster, once systematic methods are available.

Comment on "General Solution" and "Singular Solution"

Usage of the term "general solution" is not uniform in the literature. Some books use the term to mean a solution that includes *all* solutions, that is, both the particular and the singular ones. We do not adopt this definition for two reasons. First, it is frequently quite difficult to prove that a formula includes *all* solutions; hence, this definition of a general solution is rather useless in practice. Second, *linear* differential equations (satisfying rather general conditions on the coefficients) have no singular solutions (as mentioned in the text), so that for these equations a general solution as defined does include all solutions. For the latter reason, some books use the term "general solution" for linear equations only; but this seems very unfortunate.

SOLUTIONS TO PROBLEM SET 1.1, page 8

2. $y = e^{-x^2/2} + c$

4. $y = ce^{-1.5x}$

6. $y = a \cos x + b \sin x$

8. $y = -\dfrac{1}{0.2^3} e^{-0.2x} + c_1x^2 + c_2x + c_3$

10. $y = \pi e^{-2.5x^2}$

12. $y^2 - 4x^2 = 12$

14. $y = 4 - 4 \sin^2 x$

16. Substitution of $y = cx - c^2$ into the ODE gives

$$y'^2 - xy' + y = c^2 - xc + (cx - c^2) = 0.$$

Similarly,

$$y = \tfrac{1}{4}x^2, \qquad y' = \tfrac{1}{2}x, \qquad \text{thus} \qquad \tfrac{1}{4}x^2 - x(\tfrac{1}{2}x) + \tfrac{1}{4}x^2 = 0.$$

18. $e^{-3.6k} = \frac{1}{2}$, $k = 0.19254$, **(a)** $e^{-k} = 0.825$, **(b)** $3.012 \cdot 10^{-31}$.

20. k follows from $e^{18{,}000k} = \frac{1}{2}$, $k = (\ln \frac{1}{2})/18{,}000 = -0.000039$.
Answer: $e^{35{,}000k} = 0.26y_0$. Since the decay is exponential, $36{,}000 = 2 \cdot 18{,}000$ would give $(y_0/2)/2 = 0.25y_0$.

SECTION 1.2. Geometric Meaning of $y' = f(x, y)$. Direction Fields, Euler's Method, page 9

Purpose. To give the student a feel for the nature of ODEs and the general behavior of fields of solutions. This amounts to a conceptual clarification before entering into formal manipulations of solution methods, the latter being restricted to relatively small—albeit important—classes of ODEs. This approach is becoming increasingly important, especially because of the graphical power of **computer software**. It is the analog of conceptual studies of the derivative and integral in calculus as opposed to formal techniques of differentiation and integration.

Comment on Order of Sections

This section could equally well be presented later in Chap. 1, perhaps after one or two formal methods of solution have been studied.

Euler's method has been included for essentially two reasons, namely, as an early eye opener to the possibility of numerically obtaining approximate values of solutions by step-by-step computations and, secondly, to enhance the student's conceptual geometric understanding of the nature of an ODE.

Furthermore, the inaccuracy of the method will motivate the development of much more accurate methods by practically the same basic principle (in Sec. 21.1).

Problem Set 1.2 will help the student with the tasks of:

Drawing direction fields and approximate solution curves

Handling your CAS in selecting appropriate windows for specific tasks

A first look at the important Verhulst equation (Prob. 4)

Bell-shaped curves as solutions of a simple ODE

Outflow from a vessel (analytically discussed in the next section)

Discussing a few types of motion for given velocity (Parachutist, etc.)

Comparing approximate solutions for different step size

SOLUTIONS TO PROBLEM SET 1.2, page 11

2. Ellipses $x^2 + \frac{1}{4}y^2 = c$. If your CAS does not give you what you expected, change the given point.

4. Verhulst equation, to be discussed as a population model in Sec. 1.5. The given points correspond to constant solutions $[(0, 0)$ and $(0, 2)]$, an increasing solution through $(0, 1)$, and a decreasing solution through $(0, 3)$.

6. Solution $y(x) = -\arctan [1/(x + c)]$, not needed for doing the problem.

8. ODE of the bell-shaped curves $y = ce^{-x^2}$.

10. ODE of the outflow from a vessel, to be discussed in Sec. 1.3.

12. $y = 2\sqrt{t + 1}$, not needed to do the problem.

14. $y(x) = \sin (x + \frac{1}{4}\pi)$, not needed to do the problem.

16. **(a)** Your PC may give you fields of varying quality, depending on the choice of the region graphed, and good choices are often obtained only after some trial and error. Enlarging generally gives more details. Subregions where $|y'|$ is large are usually critical and often tend to give nonsense.

(b) $2x + 18yy' = 0$. Your CAS will produce the direction field well, even at points of the x-axis where the tangents of solution curves are vertical.

(c) $y^2 + x^2 = c$ (not needed for doing the problem).

(d) $y = ce^{-x/2}$ by remembering calculus.

18. $y = e^x$. The computed value for $x = 0.1$ shows that its error has decreased by about a factor 10. This is typical for this "first-order method" (Euler's method), as will be seen in Sec. 21.1.

x_n	y_n	$y(x_n)$	Error	Error in Prob. 17
0.01	1.010000	1.010050	0.000050	
0.02	1.020100	1.020201	0.000101	
0.03	1.030301	1.030455	0.000154	
0.04	1.040604	1.040811	0.000207	
0.05	1.051010	1.051271	0.000261	
0.06	1.061520	1.061837	0.000317	
0.07	1.072135	1.072508	0.000373	
0.08	1.082857	1.083287	0.000430	
0.09	1.093685	1.094174	0.000489	
0.10	1.104622	1.105171	0.000549	0.005171

20. The error is first negative, then positive, and finally decreases as the solution (which is decreasing for all positive x) approaches the limit 0. The computed values are:

x_n	y_n	Error
0	1.0000	0
0.2	1.0000	0.0000
0.4	0.9984	−0.0083
0.6	0.9729	−0.0453
0.8	0.8502	−0.0972
1.0	0.5541	−0.0541
1.2	0.2471	0.0396
1.4	0.1205	0.0363
1.6	0.0647	0.0223
1.8	0.0373	0.0130
2.0	0.0227	0.0076

SECTION 1.3. Separable ODEs. Modeling, page 12

Purpose. To familiarize the student with the first "big" method of solving ODEs, the separation of variables, and an extension of it, the reduction to separable form by a transformation of the ODE, namely, by introducing a new unknown function.

The section includes standard applications that lead to separable ODEs, namely,

1–3. Three simple separable ODEs with solutions involving $\tan x$, an exponential function, e^{-x^2} (bell-shaped curves)

4. The ODE of the exponential function, having various applications, such as in radiocarbon dating

5. A mixing problem for a single tank

6. Newton's law of cooling

7. Torricelli's law of outflow

In reducing to separability we consider

8. The transformation $u = y/x$, giving perhaps the most important reducible class of ODEs

Ince's classical book [A11] contains many further reductions as well as a systematic theory of reduction for certain classes of ODEs.

Comment on Problem 5

From the implicit solution we can get two explicit solutions

$$y = +\sqrt{c - (6x)^2}$$

representing semi-ellipses in the upper half-plane, and

$$y = -\sqrt{c - (6x)^2}$$

representing semi-ellipses in the lower half-plane. [Similarly, we can get two explicit solutions $x(y)$ representing semi-ellipses in the left and right half-planes, respectively.] On the x-axis, the tangents to the ellipses are vertical, so that $y'(x)$ does not exist. Similarly for $x'(y)$ on the y-axis.

This also illustrates that it is natural to consider solutions of ODEs on *open* rather than on *closed* intervals.

Comment on Separability

An analytic function $f(x, y)$ in a domain D of the xy-plane can be factored in D, $f(x, y) = g(x)h(y)$, if and only if in D,

$$f_{xy}f = f_x f_y$$

[D. Scott, *American Math. Monthly* **92** (1985), 422–423]. Simple cases are easy to decide, but this may save time in cases of more complicated ODEs, some of which may perhaps be of practical interest. You may perhaps ask your students to derive such a criterion.

Comments on Application

Each of those examples can be modified in various ways, for example, by changing the application or by taking another form of the tank, so that each example characterizes a whole class of applications.

The many ODEs in the problem set, much more than one would ordinarily be willing and have the time to consider, should serve to convince the student of the practical importance of ODEs; so these are ODEs to choose from, depending on the students' interest and background.

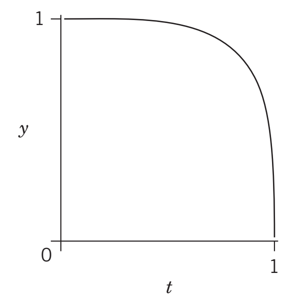
Comment on Footnote 3

Newton conceived his method of fluxions (calculus) in 1665–1666, at the age of 22. *Philosophiae Naturalis Principia Mathematica* was his most influential work.

Leibniz invented calculus independently in 1675 and introduced notations that were essential to the rapid development in this field. His first publication on differential calculus appeared in 1684.

SOLUTIONS TO PROBLEM SET 1.3, page 18

2. $y^3\,dy = -x^3\,dx$, $\tfrac{1}{4}y^4 = -\tfrac{1}{4}x^4 + \tilde{c}$, so that multiplication by 4 gives the answer $y^4 + x^4 = c$. These are curves that lie between a circle and a square, outside the circle and inside the square that touch the circle at the points of intersection with the axes. The figure shows a quarter of such a curve for $c = 1$.



Sec. 1.3. Prob. 2. Quarter of the solution curve

4. Separation, integration, and taking exponents gives

$$dy/y = \pi \cot 2\pi x\,dx, \quad \ln|y| = \tfrac{1}{2}\ln|\sin 2\pi x| + c,$$

and

$$y = c\sqrt{\sin 2\pi x}.$$

6. Separation of variables, integration, and taking the reciprocal gives

$$\frac{dy}{y^2} = e^{2x-1}\,dx, \qquad -\frac{1}{y} = \tfrac{1}{2}e^{2x-1} + \tilde{c} \qquad y = \frac{2}{c - e^{2x-1}}.$$

8. From the ODE and the suggested transformation we obtain

$$y' = v' - 4 = v^2, \qquad \text{hence} \qquad v' = v^2 + 4.$$

Separation of variables and integration gives

$$\frac{dv}{v^2 + 4} = dx \qquad \text{and} \qquad \tfrac{1}{2}\arctan\frac{v}{2} = x + \tilde{c}.$$

This implies $v = 2\tan(2x + c)$ and gives the answer

$$y = v - 4x = 2\tan(2x + c) - 4x.$$

10. From the transformation and the ODE we have

$$y' = u'x + u = 1 + \frac{y}{x} = 1 + u, \qquad \text{hence} \qquad u'x = 1.$$

Separation of variables, integration, and again using the transformation gives

$$du = dx/x, \quad u = \ln x + c, \qquad y = ux = x(\ln x + c).$$

12. Separation of variables and integration gives

$$\frac{dy}{1 + 4y^2} = dx \qquad \text{and} \qquad \tfrac{1}{2}\arctan 2y = x + \tilde{c}.$$

Hence $\arctan 2y = 2x + c$. Solving for y gives the general solution

$$y = \tfrac{1}{2}\tan(2x + c)$$

and $c = -2$ from the initial condition.

14. $\dfrac{dr}{r} = -2t\,dt$, $\ln r = -t^2 + \tilde{c}$. The general solution is $r = ce^{-t^2}$ and $c = r_0$ from the initial condition.

16. From the transformation and the ODE we have

$$y = v - x + 2 \qquad \text{and} \qquad y' = v' - 1 = v^2.$$

Hence $v' = v^2 + 1$. Separation of variables and integration gives

$$\frac{dv}{v^2 + 1} = dx \quad \text{and} \quad \arctan v = x + c \quad \text{hence} \quad v = \tan(x + c).$$

From this and the transformation we obtain

$$y = v - x + 2 = 2 - x + \tan(x + c).$$

From the initial condition we get $y(0) = 2 + \tan c = 0$ and $c = 0$, so that the answer is

$$y = \tan x - x + 2.$$

18. On the left, integrate g from y_0 to y. On the right, integrate $f(x)$ over x from x_0 to x. In Prob. 12,

$$\int_3^y w\,dw = \int_2^x (-4t)\,dt.$$

20. Let k_B and k_D be the constants of proportionality for the birth rate and death rate, respectively. Then $y' = k_B y - k_D y$, where $y(t)$ is the population at time t. By separating variables, integrating, and taking exponents,

$$dy/y = (k_B - k_D)\,dt, \qquad \ln y = (k_B - k_D)t + c^*, \qquad y = ce^{(k_B - k_D)t}.$$

22. The acceleration is $a = 9 \cdot 10^6$ meters/sec^2, and the distance traveled is 5.5 meters. This is obtained as follows. Since $s(0) = 0$ (i.e., we count time from the instant the particle enters the accelerator), we have for a motion of constant acceleration

$$\text{(A)} \qquad s(t) = a\frac{t^2}{2} + bt$$

and the velocity is

$$v(t) = s'(t) = at + b.$$

From the given data we thus obtain $v(0) = b = 10^3$ and

$$v(10^{-3}) = 10^{-3}a + 10^3 = 10^4$$

so that

$$a = 10^3(10^4 - 10^3) = 10^7 - 10^6 = 9 \cdot 10^6.$$

Finally, with this a and that b, from (A) we get

$$s(10^{-3}) = 9 \cdot 10^6 \cdot \frac{10^{-6}}{2} + 10^3 \cdot 10^{-3} = 5.5 \text{ [m]}.$$

24. Let $y(t)$ be the amount of salt in the tank at time t. Then each gallon contains $y/400$ lb of salt. $2\Delta t$ gal of water run in during a short time Δt, and $-\Delta y = 2\Delta t(y/400) = \Delta t\, y/200$ is the loss of salt during Δt. Thus $\Delta y/\Delta t = -y/200$, $y' = -0.005y$, $y(t) = 100e^{-0.005t}$. *Answer:* $y(60) = 100e^{-0.3} = 74$ [lb].

26. The model is $y' = -Ay \ln y$ with $A > 0$. Constant solutions are obtained from $y' = 0$ when $y = 0$ and 1. Between 0 and 1 the right side is positive (since $\ln y < 0$), so that the solutions grow. For $y > 1$ we have $\ln y > 0$; hence the right side is negative, so that the solutions decrease with increasing t. It follows that $y = 1$ is stable. The general solution is obtained by separation of variables, integration, and two subsequent exponentiations:

$$dy/(y \ln y) = -A\, dt, \qquad \ln(\ln y) = -At + c^*,$$
$$\ln y = ce^{-At}, \qquad y = \exp(ce^{-At}).$$

28. This follows from the inquality

$$1/2^6 = 0.016 > 0.010 > 1/2^7 = 0.0078.$$

30. Acceleration $y'' = 7t$. Hence $y' = 7t^2/2$, $y = 7t^3/6$, $y'(10) = 350$ (initial speed of further flight = end speed upon return from peak), $y(10) = 7000/6 = 1167$ (height reached after the 10 sec). At the peak, $v = 0$, $s = 0$, say; thus for the further flight (measured from the peak), $s(t) = (g/2)t^2 = 4.9t^2$, $v(t) = 9.8t = 350$ (see before). This gives the further flight time to the peak $t = t_1 = 350/9.8 = 35.7$ and the further height $s(t_1) = 4.9{t_1}^2 = 6245$, approximately. *Answer:* $1167 + 6245 = 7412$ [m].

32. $W = mg$ in Fig. 15 is the weight (the force of attraction acting on the body). Its component parallel to the surface in $mg \sin\alpha$, and $N = mg\cos\alpha$. Hence the friction is $0.2mg\cos\alpha$, and it acts against the direction of motion. From this and Newton's second law, noting that the acceleration is dv/dt (v the velocity), we obtain

$$\begin{aligned} m\frac{dv}{dt} &= mg\sin\alpha - 0.2mg\cos\alpha \\ &= m\cdot 9.80(0.500 - 0.2\cdot 0.866) \\ &= 3.203\, m. \end{aligned}$$

The mass m drops out, and two integrations give

$$v = 3.203t \qquad \text{and} \qquad s = 3.203\,\frac{t^2}{2}.$$

Since the slide is 10 meters long, the last equation with $s = 10$ gives the time

$$t = \sqrt{2\cdot 10/3.203} = 2.50.$$

From this we obtain the *answer*

$$v = 3.203\cdot 2.50 = 8.01 \text{ [meters/sec]}.$$

34. TEAM PROJECT. **(a)** Note that at the origin, $x/y = 0/0$, so that y' is undefined at the origin.

(b) $(xy)' = y + xy' = 0$, $y' = -y/x$.

(c) $y = cx$. Here the student should learn that c must not appear in the ODE. $y/x = cy'/x - y/x^2 = 0$, $y' = y/x$.

(d) The right sides $-x/y$ and y/x are the slopes of the curves. Orthogonality is important and will be discussed further in Sec. 1.6.

(e) No.

36. Team Project. B now depends on h, namely, by the Pythagorean theorem,

$$B(h) = \pi r^2 = \pi(R^2 - (R - h)^2) = \pi(2Rh - h^2).$$

Hence you can use the ODE

$$h' = -26.56(A/B)\sqrt{h}$$

in the text, with constant A as before and the new B. The latter makes further calculations different from those in Example 5.

From the given outlet size $A = 5 \text{ cm}^2$ and $B(h)$ we obtain

$$\frac{dh}{dt} = -26.56 \cdot \frac{5}{\pi(2Rh - h^2)} \sqrt{h}.$$

Now $26.56 \cdot 5/\pi = 42.27$, so that separation of variables gives

$$(2Rh^{1/2} - h^{3/2})\, dh = -42.27\, dt.$$

By integration,

$$\tfrac{4}{3}Rh^{3/2} - \tfrac{2}{5}h^{5/2} = -42.27t + c.$$

From this and the initial conditions $h(0) = R$ we obtain

$$\tfrac{4}{3}R^{5/2} - \tfrac{2}{5}R^{5/2} = 0.9333R^{5/2} = c.$$

Hence the particular solution (in implict form) is

$$\tfrac{4}{3}Rh^{3/2} - \tfrac{2}{5}h^{5/2} = -42.27t + 0.9333R^{5/2}.$$

The tank is empty ($h = 0$) for t such that

$$0 = -42.27t + 0.9333R^{5/2}; \qquad \text{hence} \qquad t = \frac{0.9333}{42.27}R^{5/2} = 0.0221R^{5/2}.$$

For $R = 1 \text{ m} = 100 \text{ cm}$ this gives

$$t = 0.0221 \cdot 100^{5/2} = 2210\ [\text{sec}] = 37\ [\text{min}].$$

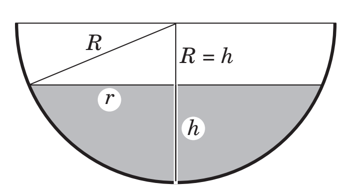
The tank has water level $R/2$ for t in the particular solution such that

$$\frac{4}{3}R\frac{R^{3/2}}{2^{3/2}} - \frac{2}{5}\frac{R^{5/2}}{2^{5/2}} = 0.9333R^{5/2} - 42.27t.$$

The left side equals $0.4007R^{5/2}$. This gives

$$t = \frac{0.4007 - 0.9333}{-42.27}R^{5/2} = 0.01260R^{5/2}.$$

For $R = 100$ this yields $t = 1260 \text{ sec} = 21 \text{ min}$. This is slightly more than half the time needed to empty the tank. This seems physically reasonable because if the water level is $R/2$, this means that $11/16$ of the total water volume has flown out, and $5/16$ is left—take into account that the velocity decreases monotone according to Torricelli's law.



Problem Set 1.3. Tank in Team Project 36

SECTION 1.4. Exact ODEs. Integrating Factors, page 20

Purpose. This is the second "big" method in this chapter, after separation of variables, and also applies to equations that are not separable. The criterion (5) is basic. Simpler cases are solved by inspection, more involved cases by integration, as explained in the text.

Comment on Condition (5)

Condition (5) is equivalent to (6″) in Sec. 10.2, which is equivalent to (6) in the case of two variables x, y. Simple connectedness of D follows from our assumptions in Sec. 1.4. Hence the differential form is exact by Theorem 3, Sec. 10.2, part (b) and part (a), in that order.

Method of Integrating Factors

This greatly increases the usefulness of solving exact equations. It is important in itself as well as in connection with linear ODEs in the next section. Problem Set 1.4 will help the student gain skill needed in finding integrating factors. Although the method has somewhat the flavor of tricks, Theorems 1 and 2 show that at least in some cases one can proceed systematically—and one of them is precisely the case needed in the next section for *linear* ODEs.

In Example 2, exactness is seen from

$$\frac{\partial}{\partial y} \cos y \sinh x + 1 = -\sin y \sinh x.$$

$$\frac{\partial}{\partial x} (-\sin y \cosh x) = -\sin y \sinh x.$$

In Example 3, separation of variables gives

$$\frac{dy}{y} = \frac{dx}{x}, \qquad y = cx.$$

SOLUTIONS TO PROBLEM SET 1.4, page 26

2. Exact, $x^4 + y^4 = c$.
Note that an ODE $f(x)\,dx + g(y)\,dy = 0$ is always exact.

4. Exact. The test gives $3e^{3\theta} = 3e^{3\theta}$. By integration,

$$u = \int e^{3\theta}\,dr = re^{3\theta} + c(\theta).$$

Hence

$$u_\theta = 3re^{3\theta} + c' = 3re^{3\theta}, \qquad c' = 0, \qquad c = \text{const}$$

6. The new ODE is

$$3(y+1)^2 x^{-4}\,dx - 2(y+1)x^{-3}\,dy = 0.$$

It is exact,

$$M_y = N_x = 6(y+1)x^{-4}.$$

The general solution is

$$(y + 1)^2 x^{-3} = c.$$

8. Exact; the test gives $-e^x \sin y$ on both sides. Integrate M with respect to x:

$$u = e^x \cos y + k(y). \qquad \text{Differentiate:} \qquad u_y = -e^x \sin y + k'.$$

Equate this to $N = -e^x \sin y$. Hence $k' = 0$, $k = \text{const.}$ *Answer:* $e^x \cos y = c$.

10. $y \cos(x + y)\,dx + [y \cos(x + y) + \sin(x + y)]\,dy = 0$ is exact because

$$\begin{aligned}[y \cos(x + y)]_y &= \cos(x + y) - y \sin(x + y)\\ &= [y \cos(x + y) + \sin(x + y)]_x.\end{aligned}$$

By inspection or systematically,

$$y \sin(x + y) = c.$$

12. $(2xye^{x^2})_y = 2xe^{x^2} = (e^{x^2})_x$ shows exactness. By integration,

$$ye^{x^2} = c.$$

$y(0) = 2$ gives $c = 2$. *Answer:* $y = 2e^{-x^2}$.

14. The integrating factor gives the exact ODE

$$(a + 1)\,x^a y^{b+1}\,dx + (b + 1)\,x^{a+1} y^b\,dy = d(x^{a+1} y^{b+1}) = 0.$$

The general solution is

$$x^{a+1} y^{b+1} = c$$

and $c = 1$ from the initial condition.

16. Team Project. (a) $e^y \cosh x = c$

(b) $R^* = \tan y$, $F = 1/\cos y$. Separation:

$$dy/\cos^2 y = -(1 + 2x)\,dx, \qquad \tan y = -x - x^2 + c.$$

(c) $R = -2/x$, $F = 1/x^2$, $x - y^2/x = c$, $v = y/x$, and separation:

$$2v\,dv/(1 - v^2) = dx/x, \qquad x^2 - y^2 = cx;$$

divide by x.

(d) Separation is simplest. $y = cx^{-3/4}$. $R = -9/(4x)$, $F(x) = x^{-9/4}$, $x^3y^4 = c$. $R^* = 3/y$, $F^*(y) = y^3$.

18. CAS Project. (a) Theorem 1 does not apply. Theorem 2 gives

$$\frac{1}{F}\frac{dF}{dy} = \frac{-1}{y^2 \sin x}(0 + 2y \sin x) = -\frac{2}{y}, \qquad F = \exp \int -\frac{2}{y}\,dy = \frac{1}{y^2}.$$

The exact ODE is

$$y^{-2}\,dy - \sin x\,dx = 0,$$

as one could have seen by inspection—any equation of the form

$$f(x)\,dx + g(y)\,dy = 0$$

is exact! We now obtain

$$u = \int -\sin x \, dx = \cos x + k(y)$$

$$u_y = k'(y) = \frac{1}{y^2}, \qquad k = -\frac{1}{y},$$

$$u = \cos x - \frac{1}{y} = c.$$

(b) Yes,

$$y' = y^2 \sin x, \qquad \frac{dy}{y^2} = \sin x \, dx, \qquad -\frac{1}{y} = -\cos x + c, \qquad y = \frac{1}{\cos x + \tilde{c}}.$$

(c) The vertical asymptotes that some CAS programs draw disturb the graph. From the solution in (b) the student should conclude that for each initial condition $y(x_0) = y_0$ with $y_0 \neq 0$ there is a unique particular solution because from (b),

$$\tilde{c} = \frac{1 - y_0 \cos x_0}{y_0}.$$

(d) $y \equiv 0$.

SECTION 1.5. Linear ODEs. Bernoulli Equation. Population Dynamics, page 27

Purpose. Linear ODEs are of great practical importance, as Problem Set 1.5 illustrates (and even more so are second-order linear ODEs in Chap. 2). We show that the homogeneous ODE of the first order is easily separated and the nonhomogeneous ODE is solved, once and for all, in the form of an integral (4) by the method of integrating factors. Of course, in simpler cases one does not need (4), as our examples illustrate.

Comment on Notation

We write

$$y' + p(x)y = r(x).$$

$p(x)$ seems standard, $r(x)$ suggests "*right side.*" The notation

$$y' + p(x)y = q(x)$$

used in some calculus books (which are not concerned with higher order ODEs) would be shortsighted here because later, in Chap. 2, we turn to second-order ODEs

$$y'' + p(x)y' + q(x)y = r(x),$$

where we need $q(x)$ on the left, thus in a quite different role (and on the right we would have to choose another letter different from that used in the first-order case).

Comment on Content

Bernoulli's equation appears occasionally in practice, so the student should remember how to handle it.

A special Bernoulli equation, the **Verhulst equation**, plays a central role in population dynamics of humans, animals, plants, and so on, and we give a short introduction to this interesting field, along with one reference in the text.

Riccati and **Clairaut equations** are less important than Bernoulli's, so we have put them in the problem set; they will not be needed in our further work.

Input and **output** have become common terms in various contexts, so we thought this a good place to mention them.

Problems 15–20 express properties that make linearity important, notably in obtaining new solutions from given ones. The counterparts of these properties will, of course, reappear in Chap. 2.

Comment on Footnote 7

Eight members of the Bernoulli family became known as mathematicians; for more details, see p. 220 in Ref. [GenRef 2] listed in App. 1.

Examples in the Text. The examples in the text concern the following.

Example 1 illustrates the use of the integral formula (4) for the linear ODE (1).

Example 2 deals with the *RL*-circuit for which the underlying physics is rather simple and straightforward and the solution exhibits exponential approach to a constant value (48/11 A). Several particular solutions are shown in Fig. 19.

Example 3 on hormone level is an input–output problem, eventually giving a periodic steady-state solution, after an exponential term has decreased to zero, theoretically as $t \to \infty$, practically after a very short time, as shown in Fig. 20.

Example 4 concerns the logistic or Verhulst ODE, perhaps the practically most important case of a Bernoulli ODE. The Bernoulli ODE is reduced to a linear ODE by setting $u = y^{1-a}$ $(a \neq 1)$, giving (10).

Example 5 concerns population dynamics, based on Malthus's and Verhulst's ODEs, both of which are autonomous. This concept is defined in connection with (13) and will be of central interest in the theory and application of systems of ODEs in Chap. 4, in particular, in Sec. 4.5 when we shall discuss the Lotka–Volterra population model.

Problem Set 1.5 stikes a balance between formal problems (3–13) for linear ODEs, experimentation (Prob. 14), some basic theory (15–21), formal problems (22–28) for nonlinear ODEs, a project (29) on transformation, two ODEs of lesser importance (Clairaut and Riccati ODEs in Team Project 30, showing singular solutions), and, finally, a variety of modeling problems (31–40) taken from various fields.

SOLUTIONS TO PROBLEM SET 1.5, page 34

4. The standard form (1) is $y' - 2y = -4x$, so that (4) gives

$$y = e^{2x}\left[\int e^{-2x}(-4x)\,dx + c\right] = ce^{2x} + 2x + 1.$$

6. From (4) with $p = 2$, $h = 2x$, $r = 4\cos 2x$ we obtain

$$y = e^{-2x}\left[\int e^{2x}\,4\cos 2x\,dx + c\right] = e^{-2x}[e^{2x}(\cos 2x + \sin 2x + c].$$

It is perhaps worthwhile mentioning that integrals of this type can more easily be evaluated by undetermined coefficients. Also, the student should verify the result by differentiation, even if it was obtained by a CAS. From the initial condition we obtain

$$y(\tfrac{1}{4}\pi) = ce^{-\pi/2} + 0 + 1 = 3; \qquad \text{hence} \qquad c = 2e^{\pi/2}.$$

The *answer* can be written

$$y = 2e^{\pi/2-2x} + \cos 2x + \sin 2x.$$

8. In (4) we have $p = \tan x$, $h = -\ln(\cos x)$, $e^h = 1/\cos x$, so that (4) gives

$$y = (\cos x)\left[\int \frac{\cos x}{\cos x}\, e^{-0.01x}\, dx + c\right] = [-100\, e^{-0.01x} + c] \cos x.$$

The initial condition gives $y(0) = -100 + c = 0$; hence $c = 100$. The particular solution is

$$y = 100(1 - e^{-0.01x}) \cos x.$$

The factor 0.01, which we include in the exponent, has the effect that the graph of y shows a long transition period. Indeed, it takes $x = 460$ to let the exponential function $e^{-0.01x}$ decrease to 0.01. Choose the x-interval of the graph accordingly.

10. The standard form (1) is

$$y' + \frac{3}{\cos^2 x}\, y = \frac{1}{\cos^2 x}.$$

Hence $h = 3 \tan x$, and (4) gives the general solution

$$y = e^{-3\tan x}\left[\int \frac{e^{3\tan x}}{\cos^2 x}\, dx + c\right].$$

To evaluate the integral, observe that the integrand is of the form

$$\tfrac{1}{3}(3\tan x)'\, e^{3\tan x};$$

that is,

$$\tfrac{1}{3}(e^{3\tan x})'.$$

Hence the integral has the value $\frac{1}{3}e^{3\tan x}$. This gives the general solution

$$y = e^{-3\tan x}[\tfrac{1}{3}e^{3\tan x} + c] = \tfrac{1}{3} + ce^{-3\tan x}.$$

The initial condition gives from this

$$y(\tfrac{1}{4}\pi) = \tfrac{1}{3} + ce^{-3} = \tfrac{4}{3}; \qquad \text{hence} \qquad c = e^3.$$

The *answer* is $y = \frac{1}{3} + e^{3-3\tan x}$.

12. $y = cx^{-4} + x^4$ is the general solution. The initial condition gives $c = 1$.

14. CAS Experiment (a) $y = x \sin(1/x) + cx$. $c = 0$ if $y(2/\pi) = 2/\pi$. y is undefined at $x = 0$, the point at which the "waves" of $\sin(1/x)$ accumulate; the factor x makes them smaller and smaller. Experiment with various x-intervals.
(b) $y = x^n[\sin(1/x) + c]$. $y(2/\pi) = (2/\pi)^n$. n need not be an integer. Try $n = \frac{1}{2}$. Try $n = -1$ and see how the "waves" near 0 become larger and larger.

16. Substitution gives the identity $0 = 0$.

These problems are of importance because they show why linear ODEs are preferable over nonlinear ones in the modeling process. Thus one favors a linear ODE over a nonlinear one if the model is a faithful mathematical representation of the problem. Furthermore, these problems illustrate the difference between homogeneous and nonhomogeneous ODEs.

18. We obtain

$$\begin{aligned}(y_1 - y_2)' + p(y_1 - y_2) &= y_1' - y_2' + py_1 - py_2\\ &= (y_1' + py_1) - (y_2' + py_2)\\ &= r - r\\ &= 0.\end{aligned}$$

20. The sum satisfies the ODE with $r_1 + r_2$ on the right. This is important as the key to the method of developing the right side into a series, then finding the solutions corresponding to single terms, and finally, adding these solutions to get a solution of the given ODE. For instance, this method is used in connection with Fourier series, as we shall see in Sec. 11.5.

22. Bernoulli equation. *First solution method:* Transformation to linear form. Set $y = 1/u$. Then $y' + y = -u'/u^2 + 1/u = 1/u^2$. Multiplication by $-u^2$ gives the linear ODE in standard form

$$u' - u = -1. \qquad \text{General solution} \qquad u = ce^x + 1.$$

Hence the given ODE has the general solution

$$y = 1/(ce^x + 1).$$

From this and the initial condition $y(0) = -\frac{1}{3}$, we obtain

$$y(0) = 1/(c + 1) = -\tfrac{1}{3}, \qquad c = -4, \qquad \textit{Answer:} \qquad y = 1/(1 - 4e^x).$$

Second solution method: Separation of variables and use of partial fractions.

$$\frac{dy}{y(y-1)} = \left(\frac{1}{y-1} - \frac{1}{y}\right) dy = dx.$$

Integration gives

$$\ln|y - 1| - \ln|y| = \ln\left|\frac{y-1}{y}\right| = x + c^*.$$

Taking exponents on both sides, we obtain

$$\frac{y-1}{y} = 1 - \frac{1}{y} = \tilde{c}e^x, \qquad \frac{1}{y} = 1 - \tilde{c}e^x, \qquad y = \frac{1}{1 + ce^x}.$$

We now continue as before.

24. $u = y^2, \quad yy' + y^2 = -x, \quad \frac{1}{2}u' + u = -x, \quad u' + 2u = -2x$; hence

$$u = e^{-2x}\left[-\int e^{2x}\, 2x\, dx + c\right] = \tfrac{1}{2} - x + ce^{-2x}, \qquad y = \sqrt{u}$$

26. This ODE can simply be solved by separating variables,

$$\cot y\, dy = dx/(x - 1), \qquad \ln|\sin y| = \ln|x - 1| + \tilde{c}$$

hence

$$y = \arcsin\,[\hat{c}(x - 1)] \qquad \text{or} \qquad x = 1 + c \sin y$$

with $c = -1$ from the initial condition.

As an alternative, we can regard it as an ODE for the unknown function $x = x(y)$ and solve it by (4) with x and y interchanged.

28. Using the given transformation $y^2 = z$, we obtain the linear ODE

$$z' + \left(1 - \frac{1}{x}\right)z = xe^x,$$

which we can solve by (4) with z instead of y,

$$z = xe^{-x}\left(\int \frac{1}{x} e^x xe^x \, dx + c\right) = xe^{-x}(\tfrac{1}{2}e^{2x} + c) = cxe^{-x} + \tfrac{1}{2}xe^x.$$

From this we obtain $y = \sqrt{z}$.

30. Team Project. **(a)** $y = Y + v$ reduces the Riccati equation to a Bernoulli equation by removing the term $h(x)$. The second transformation, $v = 1/u$, is the usual one for transforming a Bernoulli equation with y^2 on the right into a linear ODE.

Substitute $y = Y + 1/u$ into the Riccati equation to get

$$Y' - u'/u^2 + p(Y + 1/u) = g(Y^2 + 2Y/u + 1/u^2) + h.$$

Since Y is a solution, $Y' + pY = gY^2 + h$. There remains

$$-u'/u^2 + p/u = g(2Y/u + 1/u^2).$$

Multiplication by $-u^2$ gives $u' - pu = -g(2Yu + 1)$. Reshuffle terms to get

$$u' + (2Yg - p)u = -g,$$

the linear ODE as claimed.

(b) Substitute $y = Y = x$ to get $1 - 2x^4 - x = -x^4 - x^4 - x + 1$, which is true. Now substitute $y = x + 1/u$. This gives

$$1 - u'/u^2 - (2x^3 + 1)(x + 1/u) = -x^2(x^2 + 2x/u + 1/u^2) - x^4 - x + 1.$$

Most of the terms cancel on both sides. There remains $-u'/u^2 - 1/u = -x^2/u^2$. Multiplication by $-u^2$ finally gives $u' + u = x^2$. The general solution is

$$u = ce^{-x} + x^2 - 2x + 2$$

and $y = x + 1/u$. Of course, instead performing this calculation we could have used the general formula in (a), in which

$$2Yg - p = 2x(-x^2) + 2x^3 + 1 = 1 \qquad \text{and} \qquad -g = +x^2.$$

(c) By differentiation, $2y'y'' - y' - xy'' + y' = 0, y''(2y' - x) = 0$. (A) $y'' = 0$, $y = cx + a$. By substitution, $c^2 - xc + cx + a = 0, a = -c^2, y = cx - c^2$, a family of straight lines. (B) $y' = x/2, y = x^2/4 + c^*$. By substitution into the given ODE, $x^2/4 - x^2/2 + x^2/4 + c^* = 0, c^* = 0, y = x^2/4$, the envelope of the family; see Fig. 6 in Problem Set 1.1.

32. $k_1(T - T_a)$ follows from Newton's law of cooling. $k_2(T - T_w)$ models the effect of heating or cooling. $T > T_w$ calls for cooling; hence $k_2(T - T_w)$ should be negative in this case; this is true, since k_2 is assumed to be negative in this formula. Similarly for heating, when heat should be added, so that the temperature increases.

The given model is of the form

$$T' = kT + K + k_1 C \cos(\pi/12)t.$$

This can be seen by collecting terms and introducing suitable constants, $k = k_1 + k_2$ (because there are two terms involving T), and $K = -k_1 A - k_2 T_w + P$. The general solution is

$$T = ce^{kt} - K/k + L(-k\cos(\pi t/12) + (\pi/12)\sin(\pi t/12)),$$

where $L = k_1 C/(k^2 + \pi^2/144)$. The first term solves the homogeneous ODE $T' = kT$ and decreases to zero. The second term results from the constants A (in T_a), T_w, and P. The third term is sinusoidal, of period 24 hours, and time-delayed against the outside temperature, as is physically understandable.

34. $y' = ky(1 - y) = f(y)$, where $k > 0$ and y is the proportion of infected persons. Equilibrium solutions are $y = 0$ and $y = 1$. The first, $y = 0$, is unstable because $f(y) > 0$ if $0 < y < 1$ but $f(y) < 0$ for negative y. The solution $y = 1$ is stable because $f(y) > 0$ if $0 < y < 1$ and $f(y) < 0$ if $y > 1$. The general solution is

$$y = \frac{1}{1 + ce^{-kt}}.$$

It approaches 1 as $t \to \infty$. This means that eventually everybody in the population will be infected.

36. The model is

$$y' = Ay - By^2 - Hy = Ky - By^2 = y(K - By)$$

where $K = A - H$. Hence the general solution is given by (12) in Example 4 with A replaced by $K = A - H$. The equilibrium solutions are obtained from $y' = 0$; hence they are $y_1 = 0$ and $y_2 = K/B$. The population y_2 remains unchanged under harvesting, and the fraction Hy_2 of it can be harvested indefinitely—hence the name.

38. For the first 3 years you have the solution

$$y_1 = 4/(5 - 3e^{-0.8t})$$

from Prob. 36. The idea now is that, by continuity, the value $y_1(3)$ at the end of the first period is the initial value for the solution y_2 during the next period. That is,

$$y_2(3) = y_1(3) = 4/(5 - 3e^{-2.4}).$$

Now y_2 is the solution of $y' = y - y^2$ (no fishing!). Because of the initial condition this gives

$$y_2 = 4/(4 + e^{3-t} - 3e^{0.6-t}).$$

Check the continuity at $t = 3$ by calculating

$$y_2(3) = 4/(4 + e^0 - 3e^{-2.4}).$$

Similarly, for t from 6 to 9 you obtain

$$y_3 = 4/(5 - e^{4.8-0.8t} + e^{1.8-0.8t} - 3e^{-0.6-0.8t}).$$

This is a period of fishing. Check the continuity at $t = 6$:

$$y_3(6) = 4/(5 - e^0 + e^{-3} - 3e^{-5.4}).$$

This agrees with

$$y_2(6) = 4/(4 + e^{-3} - 3e^{-5.4}).$$

40. Let y denote the amount of fresh air measured in cubic feet. Then the model is obtained from the balance equation

"*Inflow minus Outflow equals the rate of change*";

that is,

$$y' = 600 - \frac{600}{20{,}000}\,y = 600 - 0.03y.$$

The general solution of this linear ODE is

$$y = ce^{-0.03t} + 20{,}000.$$

The initial condition is $y(0) = 0$ (initially no fresh air) and gives

$$y(0) = c + 20{,}000 = 0; \qquad \text{hence} \qquad c = -20{,}000.$$

The particular solution of our problem is

$$y = 20{,}000(1 - e^{-0.03t}).$$

This equals 90% if t is such that

$$e^{-0.03t} = 0.1$$

thus if $t = (\ln 0.1)/(-0.03) = 77$ [min].

SECTION 1.6. Orthogonal Trajectories. *Optional*, page 36

Purpose. To show that families of curves $F(x, y, c) = 0$ can be described by ODEs $y' = f(x, y)$ and the switch to $\tilde{y}' = -1/f(x, \tilde{y})$ produces as general solution the orthogonal trajectories. This is a nice application that may also help the student to gain more self-confidence, skill, and a deeper understanding of the nature of ODEs.

We leave this section ***optional***, for reasons of time. This will cause no gap.

The reason ODEs can be applied in this fashion results from the fact that general solutions of ODEs involve an arbitrary constant that serves as the parameter of this one-parameter family of curves determined by the given ODE, and then another general solution similarly determines the one-parameter family of the orthogonal trajectories.

Curves and their orthogonal trajectories play a role in several physical applications (e.g., in connection with electrostatic fields, fluid flows, and so on).

Problem Set 1.6 should help the student to obtain skill in representing families of curves (Probs. 1–3), finding trajectories (4–10), and understanding some basic physical and geometric applications of trajectories (11–16). This will also involve the Cauchy–Riemann equations, which are basic in complex analysis.

SOLUTIONS TO PROBLEM SET 1.6, page 38

2. $(x - c)^2 + (y - c^3)^2 - r^2 = 0$ gives a circle of radius r with center $(x_0, y_0) = (c, c^3)$ on the cubic parabola. Since this center has distance $r = \sqrt{c^2 + c^6}$, we have $r^2 = c^2 + c^6$.

4. $y' = 2x$, $\tilde{y}' = -1/(2x)$, $\tilde{y} = -\frac{1}{2}\ln x + \tilde{c}$. Note that these curves and their OTs are congruent. This is typical of ODEs $y' = f(x)$ with f not depending on y.

6. Differentiating the given formula, we obtain

$$xy' + y = 0. \qquad \text{Thus} \qquad y' = -\frac{y}{x}.$$

This is the differential equation of the given hyperbolas. Hence the differential equation of the orthogonal trajectories is

$$\tilde{y}' = \frac{x}{\tilde{y}}.$$

Separation of variables and integration gives

$$\tilde{y}\, dy = x\, dx, \qquad \tfrac{1}{2}\tilde{y}^2 = \tfrac{1}{2}x^2 + \tilde{c}.$$

Answer: The hyperbolas $x^2 - \tilde{y}^2 = c^*$ are the orthogonal trajectories of the given hyperbolas.

8. Squaring the given formula, differentiating, and solving algebraically for y', we obtain

$$y^2 - x = c, \qquad 2yy' = 1, \qquad y' = \frac{1}{2y}.$$

This is the differential equation of the given curves. Hence the differential equation of the orthogonal trajectories is

$$\tilde{y}' = -2\tilde{y}.$$

By separation of variables and integration we obtain

$$\ln |\tilde{y}| = -2x + \tilde{c}.$$

Taking exponents gives the answer

$$\tilde{y} = c^* e^{-2x}.$$

10. $x^2 + y^2 - 2cy = 0$. Solve algebraically for $2c$:

$$\frac{x^2 + y^2}{y} = \frac{x^2}{y} + y = 2c.$$

Differentiation gives

$$\frac{2x}{y} - \frac{x^2 y'}{y^2} + y' = 0.$$

By algebra,

$$y'\left(-\frac{x^2}{y^2} + 1\right) = -\frac{2x}{y}.$$

Solve for y':

$$y' = -\frac{2x}{y} \Big/ \left(\frac{y^2 - x^2}{y^2}\right) = \frac{-2xy}{y^2 - x^2}.$$

This is the ODE of the given family. Hence the ODE of the trajectories is

$$\tilde{y}' = \frac{\tilde{y}^2 - x^2}{2x\tilde{y}} = \frac{1}{2}\left(\frac{\tilde{y}}{x} - \frac{x}{\tilde{y}}\right).$$

To solve this equation, set $u = \tilde{y}/x$. Then

$$\tilde{y}' = xu' + u = \frac{1}{2}\left(u - \frac{1}{u}\right).$$

Subtract u on both sides to get

$$xu' = -\frac{u^2 + 1}{2u}.$$

Now separate variables, integrate, and take exponents, obtaining

$$\frac{2u\,du}{u^2 + 1} = -\frac{dx}{x}, \qquad \ln(u^2 + 1) = -\ln|x| + c_1, \qquad u^2 + 1 = \frac{c_2}{x}.$$

Write $u = \tilde{y}/x$ and multiply by x^2 on both sides of the last equation. This gives

$$\tilde{y}^2 + x^2 = c_2 x.$$

The *answer* is

$$(x - c_3)^2 + \tilde{y}^2 = c_3^2.$$

Note that the given circles all have their centers on the y-axis and pass through the origin. The result shows that their orthogonal trajectories are circles, too, with centers on the x-axis and passing through the origin.

12. Setting $y = 0$ gives from $x^2 + (y - c)^2 = 1 + c^2$ the equation $x^2 + c^2 = 1 + c^2$; hence $x = -1$ and $x = 1$, which verifies that those circles all pass through -1 and 1, each of them simultaneously through both points. Subtracting c^2 on both sides of the given equation, we obtain

$$x^2 + y^2 - 2cy = 1, \qquad x^2 + y^2 - 1 = 2cy, \qquad \frac{x^2 - 1}{y} + y = 2c.$$

Emphasize to your class that the ODE for the given curves must always be free of c. Having accomplished this, we can now differentiate. This gives

$$\frac{2x}{y} - \left(\frac{x^2 - 1}{y^2} - 1\right)y' = 0.$$

This is the ODE of the given curves. Replacing y' with $-1/\tilde{y}'$ and y with $\tilde{y}$, we obtain the ODE of the trajectories:

$$\frac{2x}{\tilde{y}} - \left(\frac{x^2 - 1}{\tilde{y}^2} - 1\right)\Big/(-\tilde{y}') = 0.$$

Multiplying this by $\tilde{y}'$, we get

$$\frac{2x\tilde{y}'}{\tilde{y}} + \frac{x^2 - 1}{\tilde{y}^2} - 1 = 0.$$

Multiplying this by $\tilde{y}^2/x^2$, we obtain

$$\frac{2\tilde{y}\tilde{y}'}{x} + 1 - \frac{1}{x^2} - \frac{\tilde{y}^2}{x^2} = \frac{d}{dx}\left(\frac{\tilde{y}^2}{x}\right) + 1 - \frac{1}{x^2} = 0.$$

By integration,

$$\frac{\tilde{y}^2}{x} + x + \frac{1}{x} = 2c^*. \qquad \text{Thus,} \qquad \tilde{y}^2 + x^2 + 1 = 2c^*x.$$

We see that these are the circles

$$\tilde{y}^2 + (x - c^*)^2 = c^{*2} - 1$$

dashed in Fig. 25, as claimed.

14. By differentiation,

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0, \qquad y' = -\frac{2x/a^2}{2y/b^2} = -\frac{b^2x}{a^2y}.$$

Hence the ODE of the orthogonal trajectories is

$$\tilde{y}' = \frac{a^2\tilde{y}}{b^2x}. \qquad \text{By separation,} \qquad \frac{d\tilde{y}}{\tilde{y}} = \frac{a^2}{b^2}\frac{dx}{x}.$$

Integration and taking exponents gives

$$\ln|\tilde{y}| = \frac{a^2}{b^2}\ln|x| + c^{**}, \qquad \tilde{y} = c^*x^{a^2/b^2}.$$

This shows that the ratio a^2/b^2 has substantial influence on the form of the trajectories. For $a^2 = b^2$ the given curves are circles, and we obtain straight lines as trajectories. $a^2/b^2 = 2$ gives quadratic parabolas. For higher integer values of a^2/b^2 we obtain parabolas of higher order. Intuitively, the "flatter" the ellipses are, the more rapidly the trajectories must increase to have orthogonality.

Note that our discussion also covers families of parabolas; simply interchange the roles of the curves and their trajectories.

Note further that, in the light of the present answer, our example in the text turns out to be typical.

16. $y = \int f(x)\,dx + c$. Since c is just an *additive* constant, the statement about the curves follows; these curves are obtained from any one of them by translation in the y-direction. Similarly for the OTs, whose ODE is $\tilde{y}' = -1/f(x)$ with the function on the right independent of $\tilde{y}$.

SECTION 1.7. Existence and Uniqueness of Solutions for Initial Value Problems, page 38

Purpose. To give the student at least some impression of the theory that would occupy a central position in a more theoretical course on a higher level.

Short Courses. This section can be omitted.

Comment on Iteration Methods

Iteration methods were used rather early in history, but it was Picard who made them popular. Proofs of the theorems in this section (given in books of higher level, e.g., [A11]) are based on the Picard iteration (see CAS Project 6).

Iterations are well suited for the computer because of their modest storage demand and usually short programs in which the same loop or loops are used many times, with different data. Because integration is generally not difficult for a CAS, Picard's method has gained some popularity during the past few decades.

Example 1 is simple, involving only $y = \tan x$, and is typical inasmuch as it illustrates that the actual interval of existence is much larger than the interval guaranteed by Existence Theorem 1.

Example 2 shows that IVPs violating uniqueness can be constructed relatively easily.

Lipschitz and **Hölder conditions** play a basic role in the theory of PDEs on a level substantially higher than that of our Chap.12.

SOLUTIONS TO PROBLEM SET 1.7, page 42

2. The initial condition is given at the point $x = 2$. The coefficient of y' is 0 at that point, so from the ODE we already see that something is likely to go wrong. Separating variables, integrating, and taking exponents gives

$$\frac{dy}{y} = \frac{2\,dx}{x-2}, \qquad \ln|y| = 2\ln|x-2| + c^*, \qquad y = c(x-2)^2.$$

This last expression is the general solution. It shows that $y(2) = 0$ for any c. Hence the initial condition $y(1) = 1$ cannot be satisfied. This does not contradict the theorems because we first have to write the ODE in standard form:

$$y' = f(x, y) = \frac{2y}{x-2}.$$

This shows that f is not defined when $x = 2$ (to which the initial condition refers).

4. For $k \neq 0$ we still get no solution, violating the existence as in Prob. 2. For $k = 0$ we obtain infinitely many solutions, because c remains unspecified. Thus in this case the uniqueness is violated. Neither of the two theorems is violated in either case.

6. CAS Project. **(b)** $y_n = \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^{n+1}}{(n+1)!}, \quad y = e^x - x - 1$

(c) $y_0 = 1, y_1 = 1 + 2x, y_2 = 1 + 2x + 4x^2 + \dfrac{8x^3}{3}, \cdots$

$$y(x) = \frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + \cdots$$

(d) $y = (x-1)^2, y = 0$. It approximates $y = 0$. General solution $y = (x + c)^2$.

(e) $y' = y$ would be a good candidate to begin with. Perhaps you write the initial choice as $y_0 + a$; then $a = 0$ corresponds to the choice in the text, and you see how the expressions in a are involved in the approximations. The conjecture is true for any choice of a constant (or even of a continuous function of x).

It was mentioned in footnote 10 that Picard used his iteration for proving his existence and uniqueness theorems. Since the integrations involved in the method can be handled on the computer quite efficiently, the method has gained in importance in numerics.

8. The student should get an understanding of the "intermediate" position of a Lipschitz condition between continuity and (partial) differentiability.

The student should also realize that the linear equation is basically simpler than the nonlinear one. The calculation is straightforward because we have

$$f(x, y) = r(x) - p(x)y$$

and this implies that

(A) $$f(x, y_2) - f(x, y_1) = -p(x)(y_2 - y_1).$$

This becomes a Lipschitz condition if we note that the continuity of $p(x)$ for $|x - x_0| \leqq a$ implies that $p(x)$ is bounded, say $|p(x)| \leqq M$ for all these x. Taking absolute values on both sides of (A) now gives

$$|f(x, y_2) - f(x, y_1)| \leqq M|y_2 - y_1|.$$

10. By separation and integration,

$$\frac{dy}{y} = \frac{2x - 1}{x^2 - x}\,dx, \qquad \ln|y| = \ln|x^2 - x| + c^*.$$

Taking exponents gives the general solution

$$y = c(x^2 - x).$$

From this we can see the *answers:*

No solution if $y(0) = k \neq 0$ or $y(1) = k \neq 0$.

A unique solution if $y(x_0)$ equals any y_0 and $x_0 \neq 0$ or $x_0 \neq 1$.

Infinitely many solutions if $y(0) = 0$ or $y(1) = 0$.

This does not contradict the theorems because

$$f(x, y) = \frac{2x - 1}{x^2 - x}$$

is not defined when $x = 0$ or 1.

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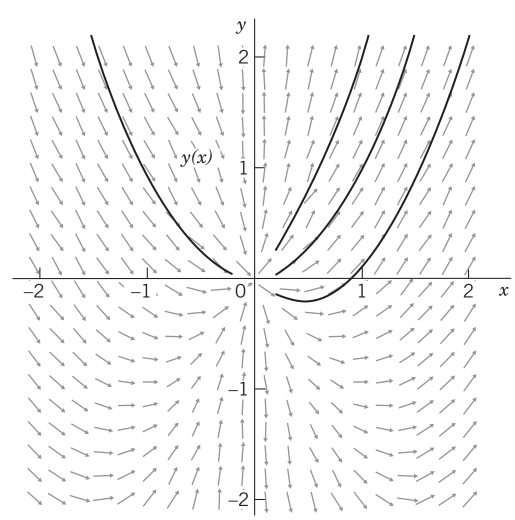
12. $y = \tanh(x + c)$. Note that the solution curves are congruent.

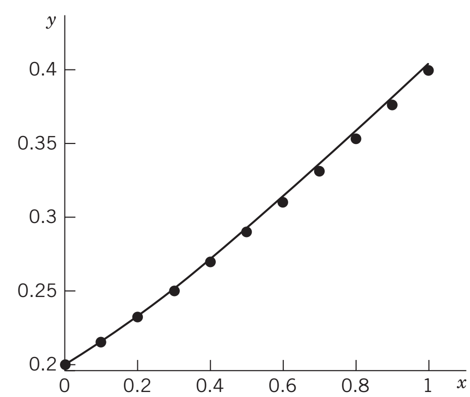
14. $y = x^2 + cx$. The figure also shows the solution curves through $(-1, 1)$ [thus, $y(-1) = 1$], $(1, 0.1)$, $(1, 1)$, and $(1, 2)$.



Problem 14. Direction field of $xy' = y + x^2$

16. Solution $y = 1/(1 + 4e^{-x})$. Computations:

x_n	y_n	Error
0	0.2000	0
0.1	0.2160	0.0005
0.2	0.2329	0.0010
0.3	0.2509	0.0015
0.4	0.2696	0.0021
0.5	0.2893	0.0026
0.6	0.3098	0.0032
0.7	0.3312	0.0037
0.8	0.3534	0.0041
0.9	0.3762	0.0046
1.0	0.3997	0.0048



Problem 16. Solution curve and computed values

18. $y = ce^{0.4x} - 25 \cos x - 10 \sin x$.

20. This Bernoulli equation (a Verhulst equation if $b < 0$) can be reduced to linear form, as shown in Example 4 of Sec. 1.5 (except for the notation). The general solution is (see (12) in Sec. 1.5)

$$y = \frac{1}{ce^{-ax} - b/a}.$$

22. The general solution of this linear differential equation is obtained as explained in Sec. 1.6,

$$y = e^{-2x^2}\left(\int e^{2x^2}e^{-2x^2}\,dx + c\right) = (x + c)e^{-2x^2}$$

From this and the initial condition $y(0) = -4.3$ we have $c = -4.3$. *Answer*:

$$y = (x - 4.3)e^{-2x^2}.$$

24. To solve this Bernoulli equation we set $u = y^{-2}$. Then $y = u^{-1/2}$, $y' = -\frac{1}{2}u^{-3/2}u'$. Substitution into the given ODE gives

$$-\tfrac{1}{2}u^{-3/2}u' + \tfrac{1}{2}u^{-1/2} = u^{-3/2}.$$

We now multiply by $-2u^{3/2}$, obtaining

$$u' - u = -2. \qquad \text{General solution:} \qquad u = ce^{x} + 2.$$

Hence

$$y = u^{-1/2} = \frac{1}{\sqrt{ce^x + 2}}.$$

From this and the initial condition $y(0) = \frac{1}{3}$ we get $c = 7$. *Answer:*

$$y = \frac{1}{\sqrt{2 + 7e^x}}.$$

26. Theorem 1 in Sec. 1.4 gives the integrating factor $F = 1/x^2$. We thus obtain the exact equation

$$\frac{1}{x} \sinh y \, dy - \frac{1}{x^2} \cosh y \, dx = 0.$$

By inspection or systematically by integration (as explained in Sec. 1.4), we obtain

$$d\left(\frac{1}{x} \cosh y\right) = 0; \qquad \text{thus,} \qquad \frac{1}{x} \cosh y = c.$$

From this and the initial condition we get $\frac{1}{3} \cdot 1 = c$. *Answer:*

$$\cosh y = \tfrac{1}{3}x.$$

28. We proceed as in Sec. 1.3. The time rate of change $y' = dy/dt$ equals the inflow of salt minus the outflow per minute.

$$y' = 20 - \frac{20}{500}\, y.$$

The initial condition is $y(0) = 80$. This gives the particular solution

$$y = 500 - 420e^{-0.04t}.$$

The limiting value is 500 lb; 95% are 475 lb, so that we get the condition

$$500 - 420e^{-0.04t} = 475,$$

from which we can determine

$$t = 25 \ln \frac{420}{25} = 70.5 \text{ [min]};$$

so it will take a little over an hour.

30. By Newton's law of cooling, since the surrounding temperature is 100°C and the initial temperature of the metal is $T(0) = 20$, we first obtain

$$T(t) = 100 - 80e^{kt}.$$

k can be determined from the condition that $T(1) = 51.5$; that is,

$$T(1) = 100 - 80e^k = 51.5,$$

so that $k = \ln(48.5/80) = -0.500$. With this value of k we can now find the time at which the metal has the temperature 99.9°C.

$$99.9 = 100 - 80e^{-0.5t}, \qquad 0.1 = 80e^{-0.5t}, \qquad t = \frac{\ln 800}{0.5} = 13.4.$$

Answer: The temperature of the metal has practically reached that of the boiling water after 13.4 min.