Chapter 1. Heat Equation

Section 1.2

1.2.9 (d) Circular cross section means that $P = 2\pi r, A = \pi r^2$, and thus $P/A = 2/r$, where r is the radius. Also $\gamma = 0$.

1.2.9 (e) $u(x,t) = u(t)$ implies that

$$c\rho \frac{du}{dt} = -\frac{2h}{r}u\ .$$

The solution of this first-order linear differential equation with constant coefficients, which satisfies the initial condition $u(0) = u_0$, is

$$u(t) = u_0 \exp\left[-\frac{2h}{c\rho r}t\right].$$

Section 1.3

1.3.2 $\partial u/\partial x$ is continuous if $K_0(x_0-) = K_0(x_0+)$, that is, if the conductivity is continuous.

Section 1.4

1.4.1 (a) Equilibrium satisfies (1.4.14), $d^2u/dx^2 = 0$, whose general solution is (1.4.17), $u = c_1 + c_2x$. The boundary condition $u(0) = 0$ implies $c_1 = 0$ and $u(L) = T$ implies $c_2 = T/L$ so that $u = Tx/L$.

1.4.1 (d) Equilibrium satisfies (1.4.14), $d^2u/dx^2 = 0$, whose general solution (1.4.17), $u = c_1 + c_2x$. From the boundary conditions, $u(0) = T$ yields $T = c_1$ and $du/dx(L) = \alpha$ yields $\alpha = c_2$. Thus $u = T + \alpha x$.

1.4.1 (f) In equilibrium, (1.2.9) becomes $d^2u/dx^2 = -Q/K_0 = -x^2$, whose general solution (by integrating twice) is $u = -x^4/12 + c_1 + c_2x$. The boundary condition $u(0) = T$ yields $c_1 = T$, while $du/dx(L) = 0$ yields $c_2 = L^3/3$. Thus $u = -x^4/12 + L^3x/3 + T$.

1.4.1 (h) Equilibrium satisfies $d^2u/dx^2 = 0$. One integration yields $du/dx = c_2$, the second integration yields the general solution $u = c_1 + c_2x$.

$$\begin{aligned} x = 0: &\quad c_2 - (c_1 - T) = 0 \\ x = L: &\quad c_2 = \alpha \text{ and thus } c_1 = T + \alpha. \end{aligned}$$

Therefore, $u = (T + \alpha) + \alpha x = T + \alpha(x + 1)$.

1.4.7 (a) For equilibrium:

$$\frac{d^2u}{dx^2} = -1 \text{ implies } u = -\frac{x^2}{2} + c_1x + c_2 \text{ and } \frac{du}{dx} = -x + c_1.$$

From the boundary conditions $\frac{du}{dx}(0) = 1$ and $\frac{du}{dx}(L) = \beta, c_1 = 1$ and $-L + c_1 = \beta$ which is consistent only if $\beta + L = 1$. If $\beta = 1 - L$, there is an equilibrium solution ($u = -\frac{x^2}{2} + x + c_2$). If $\beta \neq 1 - L$, there isn't an equilibrium solution. The difficulty is caused by the heat flow being specified at both ends and a source specified inside. An equilibrium will exist only if these three are in balance. This balance can be mathematically verified from conservation of energy:

$$\frac{d}{dt}\int_0^L c\rho u\ dx = -\frac{du}{dx}(0) + \frac{du}{dx}(L) + \int_0^L Q_0\ dx = -1 + \beta + L.$$

If $\beta + L = 1$, then the total thermal energy is constant and the initial energy = the final energy:

$$\int_0^L f(x)\ dx = \int_0^L \left(-\frac{x^2}{2} + x + c_2\right)\ dx, \quad \text{which determines} \quad c_2.$$

If $\beta + L \neq 1$, then the total thermal energy is always changing in time and an equilibrium is never reached.

Section 1.5

1.5.9 (a) In equilibrium, (1.5.14) using (1.5.19) becomes $\frac{d}{dr}\left(r\frac{du}{dr}\right) = 0$. Integrating once yields $rdu/dr = c_1$ and integrating a second time (after dividing by r) yields $u = c_1 \ln r + c_2$. An alternate general solution is $u = c_1 \ln(r/r_1) + c_3$. The boundary condition $u(r_1) = T_1$ yields $c_3 = T_1$, while $u(r_2) = T_2$ yields $c_1 = (T_2 - T_1)/\ln(r_2/r_1)$. Thus, $u = \frac{1}{\ln(r_2/r_1)}\left[(T_2 - T_1)\ln r/r_1 + T_1 \ln(r_2/r_1)\right]$.

1.5.11 For equilibrium, the radial flow at $r = a$, $2\pi a\beta$, must equal the radial flow at $r = b$, $2\pi b$. Thus $\beta = b/a$.

1.5.13 From exercise 1.5.12, in equilibrium $\frac{d}{dr}\left(r^2\frac{du}{dr}\right) = 0$. Integrating once yields $r^2 du/dr = c_1$ and integrating a second time (after dividing by r^2) yields $u = -c_1/r + c_2$. The boundary conditions $u(4) = 80$ and $u(1) = 0$ yields $80 = -c_1/4 + c_2$ and $0 = -c_1 + c_2$. Thus $c_1 = c_2 = 320/3$ or $u = \frac{320}{3}\left(1 - \frac{1}{r}\right)$.