AN INTRODUCTION TO OPTIMIZATION

SOLUTIONS MANUAL

Fourth Edition

Edwin K. P. Chong and Stanislaw H. Żak

WILEY-INTERSCIENCE

A JOHN WILEY & SONS, INC., PUBLICATION

1. Methods of Proof and Some Notation

1.1

A	B	not A	not B	A⇒B	(not B)⇒(not A)
F	F	T	T	T	T
F	T	T	F	T	T
T	F	F	T	F	F
T	T	F	F	T	T

1.2

A	B	not A	not B	A⇒B	not (A and (not B))
F	F	T	T	T	T
F	T	T	F	T	T
T	F	F	T	F	F
T	T	F	F	T	T

1.3

A	B	not (A and B)	not A	not B	(not A) or (not B))
F	F	T	T	T	T
F	T	T	T	F	T
T	F	T	F	T	T
T	T	F	F	F	F

1.4

A	B	A and B	A and (not B)	(A and B) or (A and (not B))
F	F	F	F	F
F	T	F	F	F
T	F	F	T	T
T	T	T	F	T

1.5

The cards that you should turn over are 3 and A. The remaining cards are irrelevant to ascertaining the truth or falsity of the rule. The card with S is irrelevant because S is not a vowel. The card with 8 is not relevant because the rule does not say that if a card has an even number on one side, then it has a vowel on the other side.

Turning over the A card directly verifies the rule, while turning over the 3 card verifies the contraposition.

2. Vector Spaces and Matrices

2.1

We show this by contradiction. Suppose $n < m$. Then, the number of columns of $\boldsymbol{A}$ is n. Since rank $\boldsymbol{A}$ is the maximum number of linearly independent columns of $\boldsymbol{A}$, then rank $\boldsymbol{A}$ cannot be greater than $n < m$, which contradicts the assumption that rank $\boldsymbol{A} = m$.

2.2

⇒: Since there exists a solution, then by Theorem 2.1, rank $\boldsymbol{A} = \text{rank}[\boldsymbol{A} \vdots \boldsymbol{b}]$. So, it remains to prove that rank $\boldsymbol{A} = n$. For this, suppose that rank $\boldsymbol{A} < n$ (note that it is impossible for rank $\boldsymbol{A} > n$ since $\boldsymbol{A}$ has only n columns). Hence, there exists $\boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{y} \neq \boldsymbol{0}$, such that $\boldsymbol{A}\boldsymbol{y} = \boldsymbol{0}$ (this is because the columns of

$\boldsymbol{A}$ are linearly dependent, and $\boldsymbol{Ay}$ is a linear combination of the columns of $\boldsymbol{A}$). Let $\boldsymbol{x}$ be a solution to $\boldsymbol{Ax} = \boldsymbol{b}$. Then clearly $\boldsymbol{x} + \boldsymbol{y} \neq \boldsymbol{x}$ is also a solution. This contradicts the uniqueness of the solution. Hence, $\operatorname{rank} \boldsymbol{A} = n$.

$\Leftarrow$: By Theorem 2.1, a solution exists. It remains to prove that it is unique. For this, let $\boldsymbol{x}$ and $\boldsymbol{y}$ be solutions, i.e., $\boldsymbol{Ax} = \boldsymbol{b}$ and $\boldsymbol{Ay} = \boldsymbol{b}$. Subtracting, we get $\boldsymbol{A}(\boldsymbol{x} - \boldsymbol{y}) = \boldsymbol{0}$. Since $\operatorname{rank} \boldsymbol{A} = n$ and $\boldsymbol{A}$ has n columns, then $\boldsymbol{x} - \boldsymbol{y} = \boldsymbol{0}$ and hence $\boldsymbol{x} = \boldsymbol{y}$, which shows that the solution is unique.

2.3

Consider the vectors $\bar{\boldsymbol{a}}_i = [1, \boldsymbol{a}_i^\top]^\top \in \mathbb{R}^{n+1}$, $i = 1, \ldots, k$. Since $k \geq n + 2$, then the vectors $\bar{\boldsymbol{a}}_1, \ldots, \bar{\boldsymbol{a}}_k$ must be linearly independent in $\mathbb{R}^{n+1}$. Hence, there exist $\alpha_1, \ldots \alpha_k$, not all zero, such that

$$\sum_{i=1}^{k} \alpha_i \boldsymbol{a}_i = \boldsymbol{0}.$$

The first component of the above vector equation is $\sum_{i=1}^k \alpha_i = 0$, while the last n components have the form $\sum_{i=1}^k \alpha_i \boldsymbol{a}_i = \boldsymbol{0}$, completing the proof.

2.4

a. We first postmultiply $\boldsymbol{M}$ by the matrix

$$\begin{bmatrix} \boldsymbol{I}_k & \boldsymbol{O} \\ -\boldsymbol{M}_{m-k,k} & \boldsymbol{I}_{m-k} \end{bmatrix}$$

to obtain

$$\begin{bmatrix} \boldsymbol{M}_{m-k,k} & \boldsymbol{I}_{m-k} \\ \boldsymbol{M}_{k,k} & \boldsymbol{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_k & \boldsymbol{O} \\ -\boldsymbol{M}_{m-k,k} & \boldsymbol{I}_{m-k} \end{bmatrix} = \begin{bmatrix} \boldsymbol{O} & \boldsymbol{I}_{m-k} \\ \boldsymbol{M}_{k,k} & \boldsymbol{O} \end{bmatrix}.$$

Note that the determinant of the postmultiplying matrix is 1. Next we postmultiply the resulting product by

$$\begin{bmatrix} \boldsymbol{O} & \boldsymbol{I}_k \\ \boldsymbol{I}_{m-k} & \boldsymbol{O} \end{bmatrix}$$

to obtain

$$\begin{bmatrix} \boldsymbol{O} & \boldsymbol{I}_{m-k} \\ \boldsymbol{M}_{k,k} & \boldsymbol{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{O} & \boldsymbol{I}_k \\ \boldsymbol{I}_{m-k} & \boldsymbol{O} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_k & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{M}_{k,k} \end{bmatrix}.$$

Notice that

$$\det \boldsymbol{M} = \det \left(\begin{bmatrix} \boldsymbol{I}_k & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{M}_{k,k} \end{bmatrix} \right) \det \left(\begin{bmatrix} \boldsymbol{O} & \boldsymbol{I}_k \\ \boldsymbol{I}_{m-k} & \boldsymbol{O} \end{bmatrix} \right),$$

where

$$\det \left(\begin{bmatrix} \boldsymbol{O} & \boldsymbol{I}_k \\ \boldsymbol{I}_{m-k} & \boldsymbol{O} \end{bmatrix} \right) = \pm 1.$$

The above easily follows from the fact that the determinant changes its sign if we interchange columns, as discussed in Section 2.2. Moreover,

$$\det \left(\begin{bmatrix} \boldsymbol{I}_k & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{M}_{k,k} \end{bmatrix} \right) = \det(\boldsymbol{I}_k) \det(\boldsymbol{M}_{k,k}) = \det(\boldsymbol{M}_{k,k}).$$

Hence,

$$\det \boldsymbol{M} = \pm \det \boldsymbol{M}_{k,k}.$$

b. We can see this on the following examples. We assume, without loss of generality that $\boldsymbol{M}_{m-k,k} = \boldsymbol{O}$ and let $\boldsymbol{M}_{k,k} = 2$. Thus $k = 1$. First consider the case when $m = 2$. Then we have

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{O} & \boldsymbol{I}_{m-k} \\ \boldsymbol{M}_{k,k} & \boldsymbol{O} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}.$$

Thus,

$$\det \boldsymbol{M} = -2 = \det\left(-\boldsymbol{M}_{k,k}\right).$$

Next consider the case when $m = 3$. Then

$$\det \begin{bmatrix} \boldsymbol{O} & \boldsymbol{I}_{m-k} \\ \boldsymbol{M}_{k,k} & \boldsymbol{O} \end{bmatrix} = \det \begin{bmatrix} 0 & \vdots & 1 & 0 \\ 0 & \vdots & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 2 & \vdots & 0 & 0 \end{bmatrix} = 2 \neq \det\left(-\boldsymbol{M}_{k,k}\right).$$

Therefore, in general,

$$\det \boldsymbol{M} \neq \det\left(-\boldsymbol{M}_{k,k}\right)$$

However, when $k = m/2$, that is, when all sub-matrices are square and of the same dimension, then it is true that

$$\det \boldsymbol{M} = \det\left(-\boldsymbol{M}_{k,k}\right).$$

See [121].

2.5

Let

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}$$

and suppose that each block is $k \times k$. John R. Silvester [121] showed that if at least one of the blocks is equal to $\boldsymbol{O}$ (zero matrix), then the desired formula holds. Indeed, if a row or column block is zero, then the determinant is equal to zero as follows from the determinant's properties discussed Section 2.2. That is, if $\boldsymbol{A} = \boldsymbol{B} = \boldsymbol{O}$, or $\boldsymbol{A} = \boldsymbol{C} = \boldsymbol{O}$, and so on, then obviously $\det \boldsymbol{M} = 0$. This includes the case when any three or all four block matrices are zero matrices.

If $\boldsymbol{B} = \boldsymbol{O}$ or $\boldsymbol{C} = \boldsymbol{O}$ then

$$\det \boldsymbol{M} = \det \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} = \det\left(\boldsymbol{A}\boldsymbol{D}\right).$$

The only case left to analyze is when $\boldsymbol{A} = \boldsymbol{O}$ or $\boldsymbol{D} = \boldsymbol{O}$. We will show that in either case,

$$\det \boldsymbol{M} = \det\left(-\boldsymbol{B}\boldsymbol{C}\right).$$

Without loss of generality suppose that $\boldsymbol{D} = \boldsymbol{O}$. Following arguments of John R. Silvester [121], we premultiply $\boldsymbol{M}$ by the product of three matrices whose determinants are unity:

$$\begin{bmatrix} \boldsymbol{I}_k & -\boldsymbol{I}_k \\ \boldsymbol{O} & \boldsymbol{I}_k \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_k & \boldsymbol{O} \\ \boldsymbol{I}_k & \boldsymbol{I}_k \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_k & -\boldsymbol{I}_k \\ \boldsymbol{O} & \boldsymbol{I}_k \end{bmatrix} \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{O} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{C} & \boldsymbol{O} \\ \boldsymbol{A} & \boldsymbol{B} \end{bmatrix}.$$

Hence,

$$\begin{aligned} \det \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{O} \end{bmatrix} &= \begin{bmatrix} -\boldsymbol{C} & \boldsymbol{O} \\ \boldsymbol{A} & \boldsymbol{B} \end{bmatrix} \\ &= \det\left(-\boldsymbol{C}\right) \det \boldsymbol{B} \\ &= \det\left(-\boldsymbol{I}_k\right) \det \boldsymbol{C} \det \boldsymbol{B}. \end{aligned}$$

Thus we have

$$\det \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{O} \end{bmatrix} = \det\left(-\boldsymbol{B}\boldsymbol{C}\right) = \det\left(-\boldsymbol{C}\boldsymbol{B}\right).$$

2.6

We represent the given system of equations in the form $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & -2 & 0 & -1 \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \qquad \text{and} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Using elementary row operations yields

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & -2 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -3 & -2 & -2 \end{bmatrix}, \qquad \text{and}$$

$$[\boldsymbol{A}, \boldsymbol{b}] = \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & -2 & 0 & -1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 0 & -3 & -2 & -2 & -3 \end{bmatrix},$$

from which rank $\boldsymbol{A} = 2$ and rank$[\boldsymbol{A}, \boldsymbol{b}] = 2$. Therefore, by Theorem 2.1, the system has a solution.

We next represent the system of equations as

$$\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 - 2x_3 - x_4 \\ -2 + x_4 \end{bmatrix}$$

Assigning arbitrary values to x_3 and x_4 ($x_3 = d_3$, $x_4 = d_4$), we get

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 1 - 2x_3 - x_4 \\ -2 + x_4 \end{bmatrix} \\ &= -\frac{1}{3} \begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 - 2x_3 - x_4 \\ -2 + x_4 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{4}{3}d_3 - \frac{1}{3}d_4 \\ 1 - \frac{2}{3}d_3 - \frac{2}{3}d_4 \end{bmatrix}. \end{aligned}$$

Therefore, a general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3}d_3 - \frac{1}{3}d_4 \\ 1 - \frac{2}{3}d_3 - \frac{2}{3}d_4 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} \\ -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix} d_3 + \begin{bmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ 0 \\ 1 \end{bmatrix} d_4 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

where d_3 and d_4 are arbitrary values.

2.7

1. Apply the definition of $|-a|$:

$$\begin{aligned} |-a| &= \begin{cases} -a & \text{if } -a > 0 \\ 0 & \text{if } -a = 0 \\ -(-a) & \text{if } -a < 0 \end{cases} \\ &= \begin{cases} -a & \text{if } a < 0 \\ 0 & \text{if } a = 0 \\ a & \text{if } a > 0 \end{cases} \\ &= |a|. \end{aligned}$$

2. If $a \geq 0$, then $|a| = a$. If $a < 0$, then $|a| = -a > 0 > a$. Hence $|a| \geq a$. On the other hand, $|-a| \geq -a$ (by the above). Hence, $a \geq -|-a| = -|a|$ (by property 1).

3. We have four cases to consider. First, if $a, b \geq 0$, then $a + b \geq 0$. Hence, $|a + b| = a + b = |a| + |b|$. Second, if $a, b \geq 0$, then $a + b \leq 0$. Hence $|a + b| = -(a + b) = -a - b = |a| + |b|$.
Third, if $a \geq 0$ and $b \leq 0$, then we have two further subcases:

1. If $a + b \geq 0$, then $|a + b| = a + b \leq |a| + |b|$.
2. If $a + b \leq 0$, then $|a + b| = -a - b \leq |a| + |b|$.

The fourth case, $a \leq 0$ and $b \geq 0$, is identical to the third case, with a and b interchanged.
4. We first show $|a - b| \leq |a| + |b|$. We have

$$\begin{aligned} |a - b| &= |a + (-b)| \\ &\leq |a| + |-b| \quad \text{by property 3} \\ &= |a| + |b| \quad \text{by property 1.} \end{aligned}$$

To show $||a| - |b|| \leq |a - b|$, we note that $|a| = |a - b + b| \leq |a - b| + |b|$, which implies $|a| - |b| \leq |a - b|$. On the other hand, from the above we have $|b| - |a| \leq |b - a| = |a - b|$ by property 1. Therefore, $||a| - |b|| \leq |a - b|$.
5. We have four cases. First, if $a, b \geq 0$, we have $ab \geq 0$ and hence $|ab| = ab = |a||b|$. Second, if $a, b \leq 0$, we have $ab \geq 0$ and hence $|ab| = ab = (-a)(-b) = |a||b|$. Third, if $a \leq 0$, $b \leq 0$, we have $ab \leq 0$ and hence $|ab| = -ab = a(-b) = |a||b|$. The fourth case, $a \leq 0$ and $b \geq 0$, is identical to the third case, with a and b interchanged.
6. We have

$$\begin{aligned} |a + b| &\leq |a| + |b| \quad \text{by property 3} \\ &\leq c + d. \end{aligned}$$

7. $\Rightarrow$: By property 2, $-a \leq |a|$ and $a \leq |a$. Therefore, $|a| < b$ implies $-a \leq |a| < b$ and $a \leq |a| < b$.
$\Leftarrow$: If $a \geq 0$, then $|a| = a < b$. If $a < 0$, then $|a| = -a < b$.
For the case when "$<$" is replaced by "$\leq$", we simply repeat the above proof with "$<$" replaced by "$\leq$".
8. This is simply the negation of property 7 (apply DeMorgan's Law).

2.8

Observe that we can represent $\langle \boldsymbol{x}, \boldsymbol{y} \rangle_2$ as

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle_2 = \boldsymbol{x}^\top \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \boldsymbol{y} = (\boldsymbol{Q}\boldsymbol{x})^\top (\boldsymbol{Q}\boldsymbol{y}) = \boldsymbol{x}^\top \boldsymbol{Q}^2 \boldsymbol{y},$$

where

$$\boldsymbol{Q} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Note that the matrix $\boldsymbol{Q} = \boldsymbol{Q}^\top$ is nonsingular.
1. Now, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle_2 = (\boldsymbol{Q}\boldsymbol{x})^\top (\boldsymbol{Q}\boldsymbol{x}) = \|\boldsymbol{Q}\boldsymbol{x}\|^2 \geq 0$, and

$$\begin{aligned} \langle \boldsymbol{x}, \boldsymbol{x} \rangle_2 = 0 &\Leftrightarrow \|\boldsymbol{Q}\boldsymbol{x}\|^2 = 0 \\ &\Leftrightarrow \boldsymbol{Q}\boldsymbol{x} = \boldsymbol{0} \\ &\Leftrightarrow \boldsymbol{x} = \boldsymbol{0} \end{aligned}$$

since $\boldsymbol{Q}$ is nonsingular.
2. $\langle \boldsymbol{x}, \boldsymbol{y} \rangle_2 = (\boldsymbol{Q}\boldsymbol{x})^\top (\boldsymbol{Q}\boldsymbol{y}) = (\boldsymbol{Q}\boldsymbol{y})^\top (\boldsymbol{Q}\boldsymbol{x}) = \langle \boldsymbol{y}, \boldsymbol{x} \rangle_2$.
3. We have

$$\begin{aligned} \langle \boldsymbol{x} + \boldsymbol{y}, \boldsymbol{z} \rangle_2 &= (\boldsymbol{x} + \boldsymbol{y})^\top \boldsymbol{Q}^2 \boldsymbol{z} \\ &= \boldsymbol{x}^\top \boldsymbol{Q}^2 \boldsymbol{z} + \boldsymbol{y}^\top \boldsymbol{Q}^2 \boldsymbol{z} \\ &= \langle \boldsymbol{x}, \boldsymbol{z} \rangle_2 + \langle \boldsymbol{y}, \boldsymbol{z} \rangle_2. \end{aligned}$$

4. $\langle r\boldsymbol{x}, \boldsymbol{y}\rangle_2 = (r\boldsymbol{x})^\top \boldsymbol{Q}^2\boldsymbol{y} = r\boldsymbol{x}^\top\boldsymbol{Q}^2\boldsymbol{y} = r\langle \boldsymbol{x}, \boldsymbol{y}\rangle_2$.

2.9

We have $\|\boldsymbol{x}\| = \|(\boldsymbol{x}-\boldsymbol{y})+\boldsymbol{y}\| \leq \|\boldsymbol{x}-\boldsymbol{y}\| + \|\boldsymbol{y}\|$ by the Triangle Inequality. Hence, $\|\boldsymbol{x}\| - \|\boldsymbol{y}\| \leq \|\boldsymbol{x}-\boldsymbol{y}\|$. On the other hand, from the above we have $\|\boldsymbol{y}\| - \|\boldsymbol{x}\| \leq \|\boldsymbol{y}-\boldsymbol{x}\| = \|\boldsymbol{x}-\boldsymbol{y}\|$. Combining the two inequalities, we obtain $|\|\boldsymbol{x}\| - \|\boldsymbol{y}\|| \leq \|\boldsymbol{x}-\boldsymbol{y}\|$.

2.10

Let $\epsilon > 0$ be given. Set $\delta = \epsilon$. Hence, if $\|\boldsymbol{x}-\boldsymbol{y}\| < \delta$, then by Exercise 2.9, $|\|\boldsymbol{x}\| - \|\boldsymbol{y}\|| \leq \|\boldsymbol{x}-\boldsymbol{y}\| < \delta = \epsilon$.

3. Transformations

3.1

Let $\boldsymbol{v}$ be the vector such that $\boldsymbol{x}$ are the coordinates of $\boldsymbol{v}$ with respect to $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$, and $\boldsymbol{x}'$ are the coordinates of $\boldsymbol{v}$ with respect to $\{\boldsymbol{e}_1', \boldsymbol{e}_2', \ldots, \boldsymbol{e}_n'\}$. Then,

$$\boldsymbol{v} = x_1\boldsymbol{e}_1 + \cdots + x_n\boldsymbol{e}_n = [\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n]\boldsymbol{x},$$

and

$$\boldsymbol{v} = x_1'\boldsymbol{e}_1' + \cdots + x_n'\boldsymbol{e}_n' = [\boldsymbol{e}_1', \ldots, \boldsymbol{e}_n']\boldsymbol{x}'.$$

Hence,

$$[\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n]\boldsymbol{x} = [\boldsymbol{e}_1', \ldots, \boldsymbol{e}_n']\boldsymbol{x}'$$

which implies

$$\boldsymbol{x}' = [\boldsymbol{e}_1', \ldots, \boldsymbol{e}_n']^{-1}[\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n]\boldsymbol{x} = \boldsymbol{T}\boldsymbol{x}.$$

3.2

a. We have

$$[\boldsymbol{e}_1', \boldsymbol{e}_2', \boldsymbol{e}_3'] = [\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3]\begin{bmatrix} 1 & 2 & 4 \\ 3 & -1 & 5 \\ -4 & 5 & 3 \end{bmatrix}.$$

Therefore,

$$\boldsymbol{T} = [\boldsymbol{e}_1', \boldsymbol{e}_2', \boldsymbol{e}_3']^{-1}[\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3] = \begin{bmatrix} 1 & 2 & 4 \\ 3 & -1 & 5 \\ -4 & 5 & 3 \end{bmatrix}^{-1} = \frac{1}{42}\begin{bmatrix} 28 & -14 & -14 \\ 29 & -19 & -7 \\ -11 & 13 & 7 \end{bmatrix}.$$

b. We have

$$[\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3] = [\boldsymbol{e}_1', \boldsymbol{e}_2', \boldsymbol{e}_3']\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ 3 & 4 & 5 \end{bmatrix}.$$

Therefore,

$$\boldsymbol{T} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ 3 & 4 & 5 \end{bmatrix}.$$

3.3

We have

$$[\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3] = [\boldsymbol{e}_1', \boldsymbol{e}_2', \boldsymbol{e}_3']\begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix}.$$