CHAPTER 1. LIMITS AND CONTINUITY

Section 1.1 Examples of Velocity, Growth Rate, and Area (page 63)

1. Average velocity = $\frac{\Delta x}{\Delta t} = \frac{(t+h)^2 - t^2}{h}$ m/s.

2.

h	Avg. vel. over $[2, 2+h]$
1	5.0000
0.1	4.1000
0.01	4.0100
0.001	4.0010
0.0001	4.0001

3. Guess velocity is $v = 4$ m/s at $t = 2$ s.

4. Average velocity on $[2, 2+h]$ is

$$\frac{(2+h)^2 - 4}{(2+h) - 2} = \frac{4 + 4h + h^2 - 4}{h} = \frac{4h + h^2}{h} = 4 + h.$$

As h approaches 0 this average velocity approaches 4 m/s

5. $x = 3t^2 - 12t + 1$ m at time t s.
Average velocity over interval $[1, 2]$ is
$$\frac{(3 \times 2^2 - 12 \times 2 + 1) - (3 \times 1^2 - 12 \times 1 + 1)}{2 - 1} = -3 \text{ m/s}.$$
Average velocity over interval $[2, 3]$ is
$$\frac{(3 \times 3^2 - 12 \times 3 + 1) - (3 \times 2^2 - 12 \times 2 + 1)}{3 - 2} = 3 \text{ m/s}.$$
Average velocity over interval $[1, 3]$ is
$$\frac{(3 \times 3^2 - 12 \times 3 + 1) - (3 \times 1^2 - 12 \times 1 + 1)}{3 - 1} = 0 \text{ m/s}.$$

6. Average velocity over $[t, t+h]$ is

$$\frac{3(t+h)^2 - 12(t+h) + 1 - (3t^2 - 12t + 1)}{(t+h) - t}$$
$$= \frac{6th + 3h^2 - 12h}{h} = 6t + 3h - 12 \text{ m/s}.$$

This average velocity approaches $6t - 12$ m/s as h approaches 0.
At $t = 1$ the velocity is $6 \times 1 - 12 = -6$ m/s.
At $t = 2$ the velocity is $6 \times 2 - 12 = 0$ m/s.
At $t = 3$ the velocity is $6 \times 3 - 12 = 6$ m/s.

7. At $t = 1$ the velocity is $v = -6 < 0$ so the particle is moving to the left.
At $t = 2$ the velocity is $v = 0$ so the particle is stationary.
At $t = 3$ the velocity is $v = 6 > 0$ so the particle is moving to the right.

8. Average velocity over $[t-k, t+k]$ is

$$\frac{3(t+k)^2 - 12(t+k) + 1 - [3(t-k)^2 - 12(t-k) + 1]}{(t+k) - (t-k)}$$
$$= \frac{1}{2k}\Big(3t^2 + 6tk + 3k^2 - 12t - 12k + 1 - 3t^2 + 6tk - 3k^2 + 12t - 12k + 1\Big)$$
$$= \frac{12tk - 24k}{2k} = 6t - 12 \text{ m/s},$$

which is the velocity at time t from Exercise 7.

9.

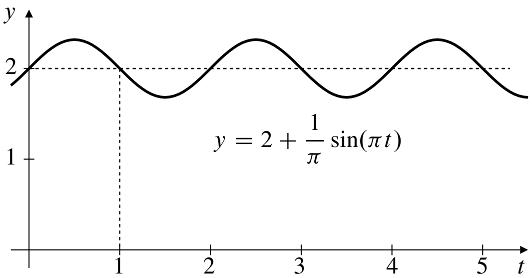

Fig. 1.1-9

At $t = 1$ the height is $y = 2$ ft and the weight is moving downward.

10. Average velocity over $[1, 1+h]$ is

$$\frac{2 + \frac{1}{\pi}\sin\pi(1+h) - \left(2 + \frac{1}{\pi}\sin\pi\right)}{h}$$
$$= \frac{\sin(\pi + \pi h)}{\pi h} = \frac{\sin\pi\cos(\pi h) + \cos\pi\sin(\pi h)}{\pi h}$$
$$= -\frac{\sin(\pi h)}{\pi h}.$$

h	Avg. vel. on $[1, 1+h]$
1.0000	0
0.1000	-0.983631643
0.0100	-0.999835515
0.0010	-0.999998355

11. The velocity at $t = 1$ is about $v = -1$ ft/s. The "–" indicates that the weight is moving downward.

12. We sketched a tangent line to the graph on page 55 in the text at $t = 20$. The line appeared to pass through the points $(10, 0)$ and $(50, 1)$. On day 20 the biomass is growing at about $(1 - 0)/(50 - 10) = 0.025$ mm^2/d.

13. The curve is steepest, and therefore the biomass is growing most rapidly, at about day 45.

14. a)

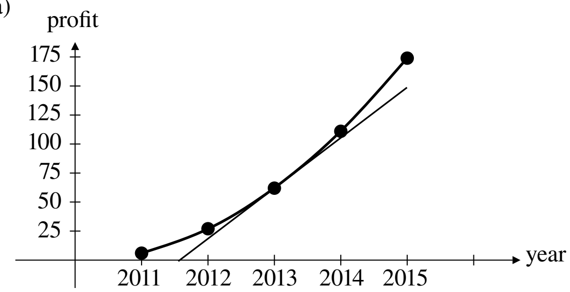

Fig. 1.1-14

b) Average rate of increase in profits between 2010 and 2012 is

$$\frac{174-62}{2012-2010} = \frac{112}{2} = 56 \text{ (thousand\$/yr)}.$$

c) Drawing a tangent line to the graph in (a) at $t = 2010$ and measuring its slope, we find that the rate of increase of profits in 2010 is about 43 thousand$/year.

Section 1.2 Limits of Functions (page 71)

1. From inspecting the graph

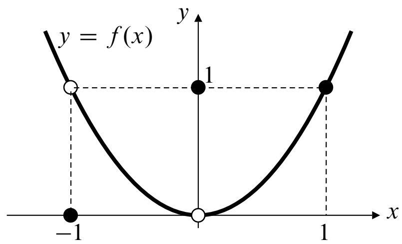

Fig. 1.2-1

we see that

$$\lim_{x\to -1} f(x) = 1, \quad \lim_{x\to 0} f(x) = 0, \quad \lim_{x\to 1} f(x) = 1.$$

2. From inspecting the graph

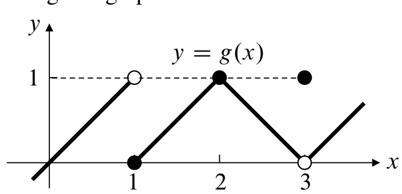

Fig. 1.2-2

we see that

$\lim_{x\to 1} g(x)$ does not exist

(left limit is 1, right limit is 0)

$$\lim_{x\to 2} g(x) = 1, \qquad \lim_{x\to 3} g(x) = 0.$$

3. $\lim_{x\to 1-} g(x) = 1$

4. $\lim_{x\to 1+} g(x) = 0$

5. $\lim_{x\to 3+} g(x) = 0$

6. $\lim_{x\to 3-} g(x) = 0$

7. $\lim_{x\to 4} (x^2 - 4x + 1) = 4^2 - 4(4) + 1 = 1$

8. $\lim_{x\to 2} 3(1-x)(2-x) = 3(-1)(2-2) = 0$

9. $\lim_{x\to 3} \frac{x+3}{x+6} = \frac{3+3}{3+6} = \frac{2}{3}$

10. $\lim_{t\to -4} \frac{t^2}{4-t} = \frac{(-4)^2}{4+4} = 2$

11. $\lim_{x\to 1} \frac{x^2-1}{x+1} = \frac{1^2-1}{1+1} = \frac{0}{2} = 0$

12. $\lim_{x\to -1} \frac{x^2-1}{x+1} = \lim_{x\to -1} (x-1) = -2$

13. $\lim_{x\to 3} \frac{x^2-6x+9}{x^2-9} = \lim_{x\to 3} \frac{(x-3)^2}{(x-3)(x+3)}$
$= \lim_{x\to 3} \frac{x-3}{x+3} = \frac{0}{6} = 0$

14. $\lim_{x\to -2} \frac{x^2+2x}{x^2-4} = \lim_{x\to -2} \frac{x}{x-2} = \frac{-2}{-4} = \frac{1}{2}$

15. $\lim_{h\to 2} \frac{1}{4-h^2}$ does not exist; denominator approaches 0 but numerator does not approach 0.

16. $\lim_{h\to 0} \frac{3h+4h^2}{h^2-h^3} = \lim_{h\to 0} \frac{3+4h}{h-h^2}$ does not exist; denominator approaches 0 but numerator does not approach 0.

17. $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} = \lim_{x\to 9} \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)}$
$= \lim_{x\to 9} \frac{x-9}{(x-9)(\sqrt{x}+3)} = \lim_{x\to 9} \frac{1}{\sqrt{x}+3} = \frac{1}{6}$

18. $\lim_{h\to 0} \frac{\sqrt{4+h}-2}{h}$
$= \lim_{h\to 0} \frac{4+h-4}{h(\sqrt{4+h}+2)}$
$= \lim_{h\to 0} \frac{1}{\sqrt{4+h}+2} = \frac{1}{4}$

19. $\lim_{x\to \pi} \frac{(x-\pi)^2}{\pi x} = \frac{0^2}{\pi^2} = 0$

20. $\lim_{x\to -2} |x-2| = |-4| = 4$

21. $\lim_{x\to 0} \frac{|x-2|}{x-2} = \frac{|-2|}{-2} = -1$

22. $\lim_{x\to 2} \frac{|x-2|}{x-2} = \lim_{x\to 2} \begin{cases} 1, & \text{if } x > 2 \\ -1, & \text{if } x < 2. \end{cases}$
Hence, $\lim_{x\to 2} \frac{|x-2|}{x-2}$ does not exist.

23. $\lim_{t\to 1} \frac{t^2-1}{t^2-2t+1}$
$\lim_{t\to 1} \frac{(t-1)(t+1)}{(t-1)^2} = \lim_{t\to 1} \frac{t+1}{t-1}$ does not exist
(denominator $\to 0$, numerator $\to 2$.)

24. $\lim_{x\to 2} \frac{\sqrt{4-4x+x^2}}{x-2}$
$= \lim_{x\to 2} \frac{|x-2|}{x-2}$ does not exist.

25. $$\lim_{t\to 0} \frac{t}{\sqrt{4+t}-\sqrt{4-t}} = \lim_{t\to 0} \frac{t(\sqrt{4+t}+\sqrt{4-t})}{(4+t)-(4-t)} = \lim_{t\to 0} \frac{\sqrt{4+t}+\sqrt{4-t}}{2} = 2$$

26. $$\lim_{x\to 1} \frac{x^2-1}{\sqrt{x+3}-2} = \lim_{x\to 1} \frac{(x-1)(x+1)(\sqrt{x+3}+2)}{(x+3)-4} = \lim_{x\to 1}(x+1)(\sqrt{x+3}+2) = (2)(\sqrt{4}+2) = 8$$

27. $$\lim_{t\to 0} \frac{t^2+3t}{(t+2)^2-(t-2)^2} = \lim_{t\to 0} \frac{t(t+3)}{t^2+4t+4-(t^2-4t+4)} = \lim_{t\to 0} \frac{t+3}{8} = \frac{3}{8}$$

28. $\lim_{s\to 0} \frac{(s+1)^2-(s-1)^2}{s} = \lim_{s\to 0} \frac{4s}{s} = 4$

29. $$\lim_{y\to 1} \frac{y-4\sqrt{y}+3}{y^2-1} = \lim_{y\to 1} \frac{(\sqrt{y}-1)(\sqrt{y}-3)}{(\sqrt{y}-1)(\sqrt{y}+1)(y+1)} = \frac{-2}{4} = \frac{-1}{2}$$

30. $$\lim_{x\to -1} \frac{x^3+1}{x+1} = \lim_{x\to -1} \frac{(x+1)(x^2-x+1)}{x+1} = 3$$

31. $$\lim_{x\to 2} \frac{x^4-16}{x^3-8} = \lim_{x\to 2} \frac{(x-2)(x+2)(x^2+4)}{(x-2)(x^2+2x+4)} = \frac{(4)(8)}{4+4+4} = \frac{8}{3}$$

32. $$\lim_{x\to 8} \frac{x^{2/3}-4}{x^{1/3}-2} = \lim_{x\to 8} \frac{(x^{1/3}-2)(x^{1/3}+2)}{(x^{1/3}-2)} = \lim_{x\to 8}(x^{1/3}+2) = 4$$

33. $$\lim_{x\to 2} \left(\frac{1}{x-2} - \frac{4}{x^2-4}\right) = \lim_{x\to 2} \frac{x+2-4}{(x-2)(x+2)} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4}$$

34. $$\lim_{x\to 2} \left(\frac{1}{x-2} - \frac{1}{x^2-4}\right) = \lim_{x\to 2} \frac{x+2-1}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x+1}{(x-2)(x+2)}$$ does not exist.

35. $$\lim_{x\to 0} \frac{\sqrt{2+x^2}-\sqrt{2-x^2}}{x^2} = \lim_{x\to 0} \frac{(2+x^2)-(2-x^2)}{x^2(\sqrt{2+x^2}+\sqrt{2-x^2})} = \lim_{x\to 0} \frac{2x^2}{x^2\left(\sqrt{2+x^2})+\sqrt{2-x^2}\right)} = \frac{2}{\sqrt{2}+\sqrt{2}} = \frac{1}{\sqrt{2}}$$

36. $$\lim_{x\to 0} \frac{|3x-1|-|3x+1|}{x} = \lim_{x\to 0} \frac{(3x-1)^2-(3x+1)^2}{x\left(|3x-1|+|3x+1|\right)} = \lim_{x\to 0} \frac{-12x}{x\left(|3x-1|+|3x+1|\right)} = \frac{-12}{1+1} = -6$$

37. $f(x) = x^2$
$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 0} \frac{2hx+h^2}{h} = \lim_{h\to 0} 2x+h = 2x$$

38. $f(x) = x^3$
$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3-x^3}{h} = \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3}{h} = \lim_{h\to 0} 3x^2+3xh+h^2 = 3x^2$$

39. $f(x) = 1/x$
$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \lim_{h\to 0} \frac{x-(x+h)}{h(x+h)x} = \lim_{h\to 0} -\frac{1}{(x+h)x} = -\frac{1}{x^2}$$

40. $f(x) = 1/x^2$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h} = \lim_{h\to 0}\frac{x^2-(x^2+2xh+h^2)}{h(x+h)^2x^2} = \lim_{h\to 0} -\frac{2x+h}{(x+h)^2x^2} = -\frac{2x}{x^4} = -\frac{2}{x^3}$$

41. $f(x) = \sqrt{x}$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

42. $f(x) = 1/\sqrt{x}$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\dfrac{1}{\sqrt{x+h}}-\dfrac{1}{\sqrt{x}}}{h} = \lim_{h\to 0}\frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} = \lim_{h\to 0}\frac{x-(x+h)}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} = \lim_{h\to 0}\frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} = \frac{-1}{2x^{3/2}}$$

43. $\lim_{x\to\pi/2}\sin x = \sin\pi/2 = 1$

44. $\lim_{x\to\pi/4}\cos x = \cos\pi/4 = 1/\sqrt{2}$

45. $\lim_{x\to\pi/3}\cos x = \cos\pi/3 = 1/2$

46. $\lim_{x\to 2\pi/3}\sin x = \sin 2\pi/3 = \sqrt{3}/2$

47.

x	$(\sin x)/x$
± 1.0	0.84147098
± 0.1	0.99833417
± 0.01	0.99998333
± 0.001	0.99999983
0.0001	1.00000000

It appears that $\lim_{x\to 0}\dfrac{\sin x}{x} = 1$.

48.

x	$(1-\cos x)/x^2$
± 1.0	0.45969769
± 0.1	0.49958347
± 0.01	0.49999583
± 0.001	0.49999996
0.0001	0.50000000

It appears that $\lim_{x\to 0}\dfrac{1-\cos x}{x^2} = \dfrac{1}{2}$.

49. $\lim_{x\to 2-}\sqrt{2-x} = 0$

50. $\lim_{x\to 2+}\sqrt{2-x}$ does not exist.

51. $\lim_{x\to -2-}\sqrt{2-x} = 2$

52. $\lim_{x\to -2+}\sqrt{2-x} = 2$

53. $\lim_{x\to 0}\sqrt{x^3-x}$ does not exist.
$(x^3 - x < 0 \text{ if } 0 < x < 1)$

54. $\lim_{x\to 0-}\sqrt{x^3-x} = 0$

55. $\lim_{x\to 0+}\sqrt{x^3-x}$ does not exist. (See # 9.)

56. $\lim_{x\to 0+}\sqrt{x^2-x^4} = 0$

57. $\lim_{x\to a-}\dfrac{|x-a|}{x^2-a^2}$

$$= \lim_{x\to a-}\frac{|x-a|}{(x-a)(x+a)} = -\frac{1}{2a} \qquad (a \neq 0)$$

58. $\lim_{x\to a+}\dfrac{|x-a|}{x^2-a^2} = \lim_{x\to a+}\dfrac{x-a}{x^2-a^2} = \dfrac{1}{2a}$

59. $\lim_{x\to 2-}\dfrac{x^2-4}{|x+2|} = \dfrac{0}{4} = 0$

60. $\lim_{x\to 2+}\dfrac{x^2-4}{|x+2|} = \dfrac{0}{4} = 0$

61. $f(x) = \begin{cases} x-1 & \text{if } x \le -1 \\ x^2+1 & \text{if } -1 < x \le 0 \\ (x+\pi)^2 & \text{if } x > 0 \end{cases}$

$\lim_{x\to -1-} f(x) = \lim_{x\to -1-} x - 1 = -1 - 1 = -2$

62. $\lim_{x\to -1+} f(x) = \lim_{x\to -1+} x^2 + 1 = 1 + 1 = 2$

63. $\lim_{x\to 0+} f(x) = \lim_{x\to 0+}(x+\pi)^2 = \pi^2$

64. $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} x^2 + 1 = 1$

65. If $\lim_{x\to 4} f(x) = 2$ and $\lim_{x\to 4} g(x) = -3$, then

a) $\lim_{x\to 4}\Big(g(x)+3\Big) = -3 + 3 = 0$

b) $\lim_{x\to 4} x f(x) = 4 \times 2 = 8$

c) $\lim_{x\to 4}\Big(g(x)\Big)^2 = (-3)^2 = 9$

d) $\lim_{x\to 4}\dfrac{g(x)}{f(x)-1} = \dfrac{-3}{2-1} = -3$

66. If $\lim x \to a\, f(x) = 4$ and $\lim_{x\to a} g(x) = -2$, then

a) $\lim_{x\to a}\Big(f(x)+g(x)\Big) = 4 + (-2) = 2$

b) $\lim_{x\to a} f(x)\cdot g(x) = 4\times(-2) = -8$

c) $\lim_{x\to a} 4g(x) = 4(-2) = -8$

d) $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{4}{-2} = -2$

67. If $\lim_{x\to 2}\dfrac{f(x)-5}{x-2} = 3$, then

$$\lim_{x\to 2}\Big(f(x)-5\Big) = \lim_{x\to 2}\frac{f(x)-5}{x-2}(x-2) = 3(2-2) = 0.$$

Thus $\lim_{x\to 2} f(x) = 5$.

68. If $\lim_{x\to 0}\dfrac{f(x)}{x^2} = -2$ then
$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2\dfrac{f(x)}{x^2} = 0\times(-2) = 0$,
and similarly, $\lim_{x\to 0}\dfrac{f(x)}{x} = \lim_{x\to 0} x\,\dfrac{f(x)}{x^2} = 0\times(-2) = 0$.

69.

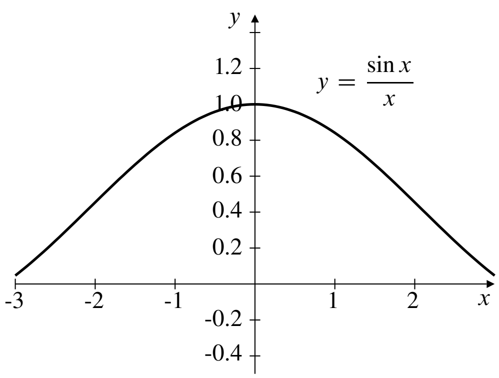

Fig. 1.2-69

$\lim_{x\to 0}\dfrac{\sin x}{x} = 1$

70.

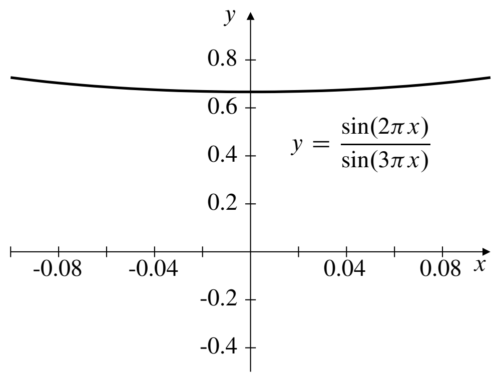

Fig. 1.2-70

$\lim_{x\to 0}\sin(2\pi x)/\sin(3\pi x) = 2/3$

71.

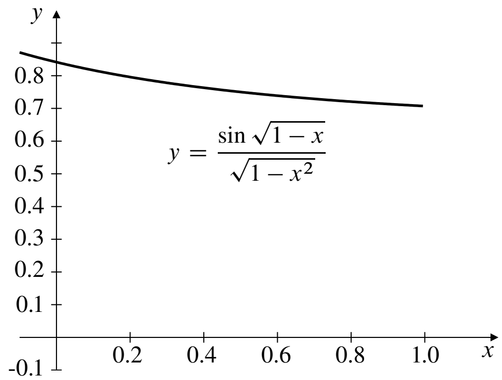

Fig. 1.2-71

$\lim_{x\to 1-}\dfrac{\sin\sqrt{1-x}}{\sqrt{1-x^2}} \approx 0.7071$

72.

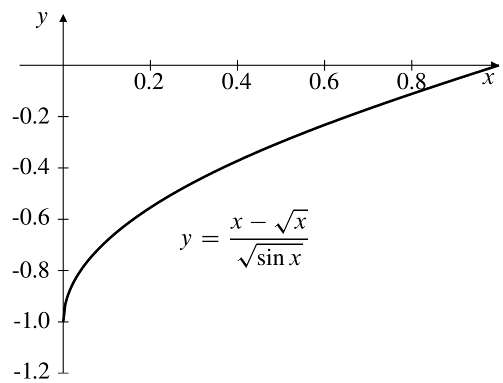

Fig. 1.2-72

$\lim_{x\to 0+}\dfrac{x-\sqrt{x}}{\sqrt{\sin x}} = -1$

73.

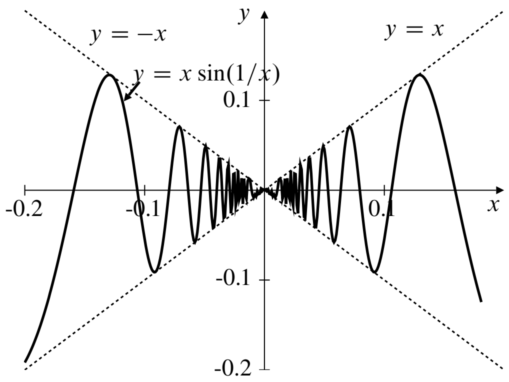

Fig. 1.2-73

$f(x) = x\sin(1/x)$ oscillates infinitely often as x approaches 0, but the amplitude of the oscillations decreases and, in fact, $\lim_{x\to 0} f(x) = 0$. This is predictable because $|x\sin(1/x)| \le |x|$. (See Exercise 95 below.)

74. Since $\sqrt{5-2x^2} \le f(x) \le \sqrt{5-x^2}$ for $-1 \le x \le 1$, and $\lim_{x\to 0}\sqrt{5-2x^2} = \lim_{x\to 0}\sqrt{5-x^2} = \sqrt{5}$, we have $\lim_{x\to 0} f(x) = \sqrt{5}$ by the squeeze theorem.

75. Since $2 - x^2 \le g(x) \le 2\cos x$ for all x, and since $\lim_{x\to 0}(2 - x^2) = \lim_{x\to 0} 2\cos x = 2$, we have $\lim_{x\to 0} g(x) = 2$ by the squeeze theorem.

76. a)

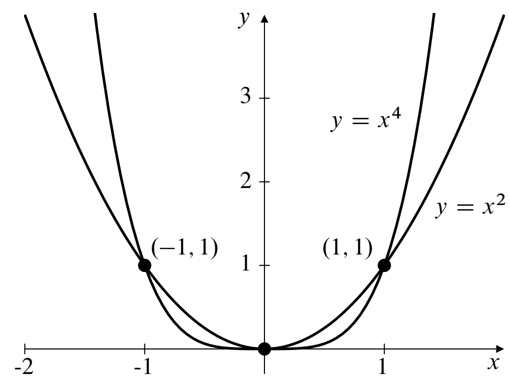

Fig. 1.2-76

b) Since the graph of f lies between those of x^2 and x^4, and since these latter graphs come together at $(\pm 1, 1)$ and at $(0, 0)$, we have $\lim_{x\to\pm 1} f(x) = 1$ and $\lim_{x\to 0} f(x) = 0$ by the squeeze theorem.

77. $x^{1/3} < x^3$ on $(-1, 0)$ and $(1, \infty)$. $x^{1/3} > x^3$ on $(-\infty, -1)$ and $(0, 1)$. The graphs of $x^{1/3}$ and x^3 intersect at $(-1, -1)$, $(0, 0)$, and $(1, 1)$. If the graph of $h(x)$ lies between those of $x^{1/3}$ and x^3, then we can determine $\lim_{x\to a} h(x)$ for $a = -1$, $a = 0$, and $a = 1$ by the squeeze theorem. In fact

$$\lim_{x\to -1} h(x) = -1, \quad \lim_{x\to 0} h(x) = 0, \quad \lim_{x\to 1} h(x) = 1.$$

78. $f(x) = s\sin\dfrac{1}{x}$ is defined for all $x \ne 0$; its domain is $(-\infty, 0) \cup (0, \infty)$. Since $|\sin t| \le 1$ for all t, we have $|f(x)| \le |x|$ and $-|x| \le f(x) \le |x|$ for all $x \ne 0$. Since $\lim_{x\to 0} = (-|x|) = 0 = \lim_{x\to 0} |x|$, we have $\lim_{x\to 0} f(x) = 0$ by the squeeze theorem.

79. $|f(x)| \le g(x) \Rightarrow -g(x) \le f(x) \le g(x)$
Since $\lim_{x\to a} g(x) = 0$, therefore $0 \le \lim_{x\to a} f(x) \le 0$.
Hence, $\lim_{x\to a} f(x) = 0$.
If $\lim_{x\to a} g(x) = 3$, then either $-3 \le \lim_{x\to a} f(x) \le 3$ or $\lim_{x\to a} f(x)$ does not exist.

Section 1.3 Limits at Infinity and Infinite Limits (page 78)

1. $\displaystyle\lim_{x\to\infty}\frac{x}{2x-3} = \lim_{x\to\infty}\frac{1}{2-(3/x)} = \frac{1}{2}$

2. $\displaystyle\lim_{x\to\infty}\frac{x}{x^2-4} = \lim_{x\to\infty}\frac{1/x}{1-(4/x^2)} = \frac{0}{1} = 0$

3. $\displaystyle\lim_{x\to\infty}\frac{3x^3-5x^2+7}{8+2x-5x^3}$

$$= \lim_{x\to\infty}\frac{3-\dfrac{5}{x}+\dfrac{7}{x^3}}{\dfrac{8}{x^3}+\dfrac{2}{x^2}-5} = -\frac{3}{5}$$

4. $\displaystyle\lim_{x\to-\infty}\frac{x^2-2}{x-x^2}$

$$= \lim_{x\to-\infty}\frac{1-\dfrac{2}{x^2}}{\dfrac{1}{x}-1} = \frac{1}{-1} = -1$$

5. $\displaystyle\lim_{x\to-\infty}\frac{x^2+3}{x^3+2} = \lim_{x\to-\infty}\frac{\dfrac{1}{x}+\dfrac{3}{x^3}}{1+\dfrac{2}{x^3}} = 0$

6. $\displaystyle\lim_{x\to\infty}\frac{x^2+\sin x}{x^2+\cos x} = \lim_{x\to\infty}\frac{1+\dfrac{\sin x}{x^2}}{1+\dfrac{\cos x}{x^2}} = \frac{1}{1} = 1$

We have used the fact that $\lim_{x\to\infty}\dfrac{\sin x}{x^2} = 0$ (and similarly for cosine) because the numerator is bounded while the denominator grows large.

7. $\displaystyle\lim_{x\to\infty}\frac{3x+2\sqrt{x}}{1-x}$

$$= \lim_{x\to\infty}\frac{3+\dfrac{2}{\sqrt{x}}}{\dfrac{1}{x}-1} = -3$$

8. $\lim_{x\to\infty} \frac{2x-1}{\sqrt{3x^2+x+1}}$

$= \lim_{x\to\infty} \frac{x\left(2-\frac{1}{x}\right)}{|x|\sqrt{3+\frac{1}{x}+\frac{1}{x^2}}}$ (but $|x| = x$ as $x \to \infty$)

$= \lim_{x\to\infty} \frac{2-\frac{1}{x}}{\sqrt{3+\frac{1}{x}+\frac{1}{x^2}}} = \frac{2}{\sqrt{3}}$

9. $\lim_{x\to-\infty} \frac{2x-1}{\sqrt{3x^2+x+1}}$

$= \lim_{x\to-\infty} \frac{2-\frac{1}{x}}{-\sqrt{3+\frac{1}{x}+\frac{1}{x^2}}} = -\frac{2}{\sqrt{3}}$,

because $x \to -\infty$ implies that $x < 0$ and so $\sqrt{x^2} = -x$.

10. $\lim_{x\to-\infty} \frac{2x-5}{|3x+2|} = \lim_{x\to-\infty} \frac{2x-5}{-(3x+2)} = -\frac{2}{3}$

11. $\lim_{x\to 3} \frac{1}{3-x}$ does not exist.

12. $\lim_{x\to 3} \frac{1}{(3-x)^2} = \infty$

13. $\lim_{x\to 3-} \frac{1}{3-x} = \infty$

14. $\lim_{x\to 3+} \frac{1}{3-x} = -\infty$

15. $\lim_{x\to -5/2} \frac{2x+5}{5x+2} = \frac{0}{\frac{-25}{2}+2} = 0$

16. $\lim_{x\to -2/5} \frac{2x+5}{5x+2}$ does not exist.

17. $\lim_{x\to -(2/5)-} \frac{2x+5}{5x+2} = -\infty$

18. $\lim_{x\to -2/5+} \frac{2x+5}{5x+2} = \infty$

19. $\lim_{x\to 2+} \frac{x}{(2-x)^3} = -\infty$

20. $\lim_{x\to 1-} \frac{x}{\sqrt{1-x^2}} = \infty$

21. $\lim_{x\to 1+} \frac{1}{|x-1|} = \infty$

22. $\lim_{x\to 1-} \frac{1}{|x-1|} = \infty$

23. $\lim_{x\to 2} \frac{x-3}{x^2-4x+4} = \lim_{x\to 2} \frac{x-3}{(x-2)^2} = -\infty$

24. $\lim_{x\to 1+} \frac{\sqrt{x^2-x}}{x-x^2} = \lim_{x\to 1+} \frac{-1}{\sqrt{x^2-x}} = -\infty$

25. $\lim_{x\to\infty} \frac{x+x^3+x^5}{1+x^2+x^3}$

$= \lim_{x\to\infty} \frac{\frac{1}{x^2}+1+x^2}{\frac{1}{x^3}+\frac{1}{x}+1} = \infty$

26. $\lim_{x\to\infty} \frac{x^3+3}{x^2+2} = \lim_{x\to\infty} \frac{x+\frac{3}{x^2}}{1+\frac{2}{x^2}} = \infty$

27. $\lim_{x\to\infty} \frac{x\sqrt{x+1}\left(1-\sqrt{2x+3}\right)}{7-6x+4x^2}$

$= \lim_{x\to\infty} \frac{x^2\left(\sqrt{1+\frac{1}{x}}\right)\left(\frac{1}{\sqrt{x}}-\sqrt{2+\frac{3}{x}}\right)}{x^2\left(\frac{7}{x^2}-\frac{6}{x}+4\right)}$

$= \frac{1(-\sqrt{2})}{4} = -\frac{1}{4}\sqrt{2}$

28. $\lim_{x\to\infty} \left(\frac{x^2}{x+1} - \frac{x^2}{x-1}\right) = \lim_{x\to\infty} \frac{-2x^2}{x^2-1} = -2$

29. $\lim_{x\to-\infty} \left(\sqrt{x^2+2x} - \sqrt{x^2-2x}\right)$

$= \lim_{x\to-\infty} \frac{(x^2+2x)-(x^2-2x)}{\sqrt{x^2+2x}+\sqrt{x^2-2x}}$

$= \lim_{x\to-\infty} \frac{4x}{(-x)\left(\sqrt{1+\frac{2}{x}}+\sqrt{1-\frac{2}{x}}\right)}$

$= -\frac{4}{1+1} = -2$

30. $\lim_{x\to\infty} \left(\sqrt{x^2+2x} - \sqrt{x^2-2x}\right)$

$= \lim_{x\to\infty} \frac{x^2+2x-x^2+2x}{\sqrt{x^2+2x}+\sqrt{x^2-2x}}$

$= \lim_{x\to\infty} \frac{4x}{x\sqrt{1+\frac{2}{x}}+x\sqrt{1-\frac{2}{x}}}$

$= \lim_{x\to\infty} \frac{4}{\sqrt{1+\frac{2}{x}}+\sqrt{1-\frac{2}{x}}} = \frac{4}{2} = 2$

31. $\lim_{x\to\infty} \frac{1}{\sqrt{x^2-2x}-x}$

$= \lim_{x\to\infty} \frac{\sqrt{x^2-2x}+x}{(\sqrt{x^2-2x}+x)(\sqrt{x^2-2x}-x)}$

$= \lim_{x\to\infty} \frac{\sqrt{x^2-2x}+x}{x^2-2x-x^2}$

$= \lim_{x\to\infty} \frac{x(\sqrt{1-(2/x)}+1)}{-2x} = \frac{2}{-2} = -1$

32. $\lim_{x\to-\infty} \frac{1}{\sqrt{x^2+2x}-x} = \lim_{x\to-\infty} \frac{1}{|x|(\sqrt{1+(2/x)}+1} = 0$

33. By Exercise 35, $y = -1$ is a horizontal asymptote (at the right) of $y = \dfrac{1}{\sqrt{x^2 - 2x} - x}$. Since

$$\lim_{x\to-\infty} \frac{1}{\sqrt{x^2 - 2x} - x} = \lim_{x\to-\infty} \frac{1}{|x|(\sqrt{1 - (2/x)} + 1} = 0,$$

$y = 0$ is also a horizontal asymptote (at the left).
Now $\sqrt{x^2 - 2x} - x = 0$ if and only if $x^2 - 2x = x^2$, that is, if and only if $x = 0$. The given function is undefined at $x = 0$, and where $x^2 - 2x < 0$, that is, on the interval $[0, 2]$. Its only vertical asymptote is at $x = 0$, where $\lim_{x\to 0-} \dfrac{1}{\sqrt{x^2 - 2x} - x} = \infty$.

34. Since $\displaystyle\lim_{x\to\infty} \frac{2x-5}{|3x+2|} = \frac{2}{3}$ and $\displaystyle\lim_{x\to-\infty} \frac{2x-5}{|3x+2|} = -\frac{2}{3}$, $y = \pm(2/3)$ are horizontal asymptotes of $y = (2x - 5)/|3x + 2|$. The only vertical asymptote is $x = -2/3$, which makes the denominator zero.

35. $\displaystyle\lim_{x\to 0+} f(x) = 1$

36. $\displaystyle\lim_{x\to 1} f(x) = \infty$

37.

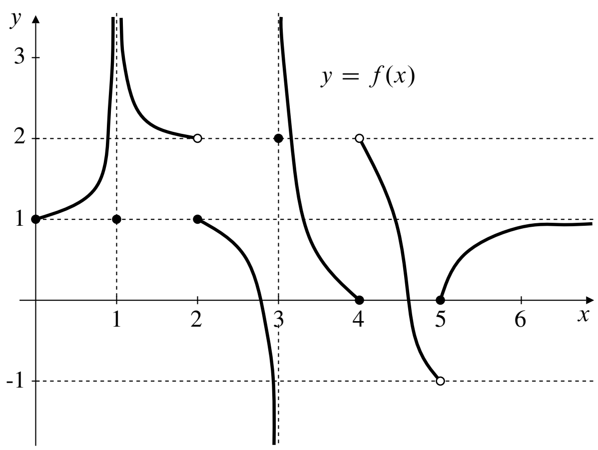

Fig. 1.3-37

$\lim_{x\to 2+} f(x) = 1$

38. $\displaystyle\lim_{x\to 2-} f(x) = 2$

39. $\displaystyle\lim_{x\to 3-} f(x) = -\infty$

40. $\displaystyle\lim_{x\to 3+} f(x) = \infty$

41. $\displaystyle\lim_{x\to 4+} f(x) = 2$

42. $\displaystyle\lim_{x\to 4-} f(x) = 0$

43. $\displaystyle\lim_{x\to 5-} f(x) = -1$

44. $\displaystyle\lim_{x\to 5+} f(x) = 0$

45. $\displaystyle\lim_{x\to\infty} f(x) = 1$

46. horizontal: $y = 1$; vertical: $x = 1$, $x = 3$.

47. $\displaystyle\lim_{x\to 3+} \lfloor x \rfloor = 3$

48. $\displaystyle\lim_{x\to 3-} \lfloor x \rfloor = 2$

49. $\displaystyle\lim_{x\to 3} \lfloor x \rfloor$ does not exist

50. $\displaystyle\lim_{x\to 2.5} \lfloor x \rfloor = 2$

51. $\displaystyle\lim_{x\to 0+} \lfloor 2 - x \rfloor = \lim_{x\to 2-} \lfloor x \rfloor = 1$

52. $\displaystyle\lim_{x\to -3-} \lfloor x \rfloor = -4$

53. $\displaystyle\lim_{t\to t_0} C(t) = C(t_0)$ except at integers t_0
$\displaystyle\lim_{t\to t_0-} C(t) = C(t_0)$ everywhere
$\displaystyle\lim_{t\to t_0+} C(t) = C(t_0)$ if $t_0 \neq$ an integer
$\displaystyle\lim_{t\to t_0+} C(t) = C(t_0) + 1.5$ if t_0 is an integer

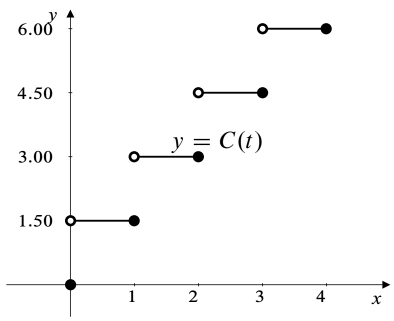

Fig. 1.3-53

54. $\displaystyle\lim_{x\to 0+} f(x) = L$
(a) If f is even, then $f(-x) = f(x)$.
Hence, $\displaystyle\lim_{x\to 0-} f(x) = L$.
(b) If f is odd, then $f(-x) = -f(x)$.
Therefore, $\displaystyle\lim_{x\to 0-} f(x) = -L$.

55. $\displaystyle\lim_{x\to 0+} f(x) = A$, $\displaystyle\lim_{x\to 0-} f(x) = B$

a) $\displaystyle\lim_{x\to 0+} f(x^3 - x) = B$ (since $x^3 - x < 0$ if $0 < x < 1$)

b) $\displaystyle\lim_{x\to 0-} f(x^3 - x) = A$ (because $x^3 - x > 0$ if $-1 < x < 0$)

c) $\displaystyle\lim_{x\to 0-} f(x^2 - x^4) = A$

d) $\displaystyle\lim_{x\to 0+} f(x^2 - x^4) = A$ (since $x^2 - x^4 > 0$ for $0 < |x| < 1$)

Section 1.4 Continuity (page 87)

1. g is continuous at $x = -2$, discontinuous at $x = -1,\ 0, 1,$ and 2. It is left continuous at $x = 0$ and right continuous at $x = 1$.

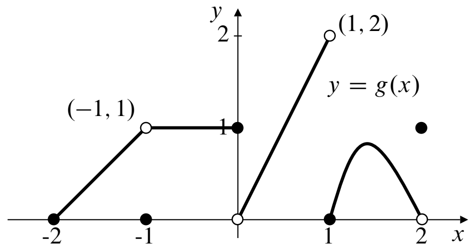

Fig. 1.4-1

2. g has removable discontinuities at $x = -1$ and $x = 2$. Redefine $g(-1) = 1$ and $g(2) = 0$ to make g continuous at those points.

3. g has no absolute maximum value on $[-2, 2]$. It takes on every positive real value less than 2, but does not take the value 2. It has absolute minimum value 0 on that interval, assuming this value at the three points $x = -2$, $x = -1$, and $x = 1$.

4. Function f is discontinuous at $x = 1,\ 2,\ 3,\ 4,$ and 5. f is left continuous at $x = 4$ and right continuous at $x = 2$ and $x = 5$.

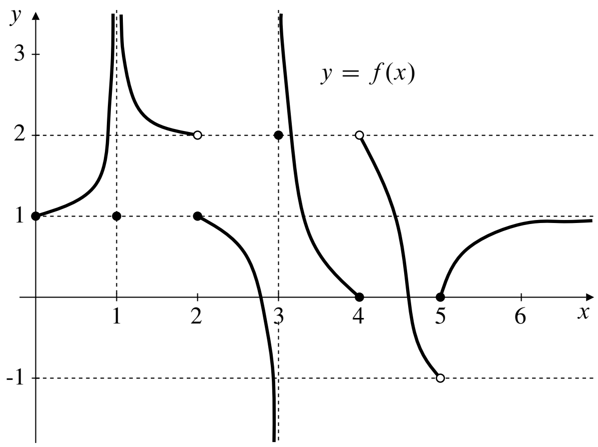

Fig. 1.4-4

5. f cannot be redefined at $x = 1$ to become continuous there because $\lim_{x\to 1} f(x)$ $(= \infty)$ does not exist. (∞ is not a real number.)

6. $\operatorname{sgn} x$ is not defined at $x = 0$, so cannot be either continuous or discontinuous there. (Functions can be continuous or discontinuous only at points in their domains!)

7. $f(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } x \ge 0 \end{cases}$ is continuous everywhere on the real line, even at $x = 0$ where its left and right limits are both 0, which is $f(0)$.

8. $f(x) = \begin{cases} x & \text{if } x < -1 \\ x^2 & \text{if } x \ge -1 \end{cases}$ is continuous everywhere on the real line except at $x = -1$ where it is right continuous, but not left continuous.

$$\begin{aligned} \lim_{x\to -1-} f(x) &= \lim_{x\to -1-} x = -1 \ne 1 \\ &= f(-1) = \lim_{x\to -1+} x^2 = \lim_{x\to -1+} f(x). \end{aligned}$$

9. $f(x) = \begin{cases} 1/x^2 & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$ is continuous everywhere except at $x = 0$, where it is neither left nor right continuous since it does not have a real limit there.

10. $f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ 0.987 & \text{if } x > 1 \end{cases}$ is continuous everywhere except at $x = 1$, where it is left continuous but not right continuous because $0.987 \ne 1$. Close, as they say, but no cigar.

11. The least integer function $\lceil x \rceil$ is continuous everywhere on $\mathbb{R}$ except at the integers, where it is left continuous but not right continuous.

12. $C(t)$ is discontinuous only at the integers. It is continuous on the left at the integers, but not on the right.

13. Since $\dfrac{x^2-4}{x-2} = x + 2$ for $x \ne 2$, we can define the function to be $2 + 2 = 4$ at $x = 2$ to make it continuous there. The continuous extension is $x + 2$.

14. Since $\dfrac{1+t^3}{1-t^2} = \dfrac{(1+t)(1-t+t^2)}{(1+t)(1-t)} = \dfrac{1-t+t^2}{1-t}$ for $t \ne -1$, we can define the function to be $3/2$ at $t = -1$ to make it continuous there. The continuous extension is $\dfrac{1-t+t^2}{1-t}$.

15. Since $\dfrac{t^2-5t+6}{t^2-t-6} = \dfrac{(t-2)(t-3)}{(t+2)(t-3)} = \dfrac{t-2}{t+2}$ for $t \ne 3$, we can define the function to be $1/5$ at $t = 3$ to make it continuous there. The continuous extension is $\dfrac{t-2}{t+2}$.

16. Since

$$\frac{x^2-2}{x^4-4} = \frac{(x-\sqrt{2})(x+\sqrt{2})}{(x-\sqrt{2})(x+\sqrt{2})(x^2+2)} = \frac{x+\sqrt{2}}{(x+\sqrt{2})(x^2+2)}$$

for $x \ne \sqrt{2}$, we can define the function to be $1/4$ at $x = \sqrt{2}$ to make it continuous there. The continuous extension is $\dfrac{x+\sqrt{2}}{(x+\sqrt{2})(x^2+2)}$. (Note: cancelling the $x + \sqrt{2}$ factors provides a further continuous extension to $x = -\sqrt{2}$.

17. $\lim_{x\to 2+} f(x) = k - 4$ and $\lim_{x\to 2-} f(x) = 4 = f(2)$. Thus f will be continuous at $x = 2$ if $k - 4 = 4$, that is, if $k = 8$.

18. $\lim_{x\to 3-} g(x) = 3 - m$ and $\lim_{x\to 3+} g(x) = 1 - 3m = g(3)$. Thus g will be continuous at $x = 3$ if $3 - m = 1 - 3m$, that is, if $m = -1$.

19. x^2 has no maximum value on $-1 < x < 1$; it takes all positive real values less than 1, but it does not take the value 1. It does have a minimum value, namely 0 taken on at $x = 0$.

20. The Max-Min Theorem says that a continuous function defined on a closed, finite interval must have maximum and minimum values. It does not say that other functions cannot have such values. The Heaviside function is not continuous on $[-1, 1]$ (because it is discontinuous at $x = 0$), but it still has maximum and minimum values. Do not confuse a theorem with its converse.

21. Let the numbers be x and y, where $x \geq 0$, $y \geq 0$, and $x + y = 8$. If P is the product of the numbers, then

$$P = xy = x(8 - x) = 8x - x^2 = 16 - (x - 4)^2.$$

Therefore $P \leq 16$, so P is bounded. Clearly $P = 16$ if $x = y = 4$, so the largest value of P is 16.

22. Let the numbers be x and y, where $x \geq 0$, $y \geq 0$, and $x + y = 8$. If S is the sum of their squares then

$$\begin{aligned} S &= x^2 + y^2 = x^2 + (8 - x)^2 \\ &= 2x^2 - 16x + 64 = 2(x - 4)^2 + 32. \end{aligned}$$

Since $0 \leq x \leq 8$, the maximum value of S occurs at $x = 0$ or $x = 8$, and is 64. The minimum value occurs at $x = 4$ and is 32.

23. Since $T = 100 - 30x + 3x^2 = 3(x - 5)^2 + 25$, T will be minimum when $x = 5$. Five programmers should be assigned, and the project will be completed in 25 days.

24. If x desks are shipped, the shipping cost per desk is

$$\begin{aligned} C = \frac{245x - 30x^2 + x^3}{x} &= x^2 - 30x + 245 \\ &= (x - 15)^2 + 20. \end{aligned}$$

This cost is minimized if $x = 15$. The manufacturer should send 15 desks in each shipment, and the shipping cost will then be \$20 per desk.

25. $f(x) = \dfrac{x^2 - 1}{x} = \dfrac{(x - 1)(x + 1)}{x}$
$f = 0$ at $x = \pm 1$. f is not defined at 0.
$f(x) > 0$ on $(-1, 0)$ and $(1, \infty)$.
$f(x) < 0$ on $(-\infty, -1)$ and $(0, 1)$.

26. $f(x) = x^2 + 4x + 3 = (x + 1)(x + 3)$
$f(x) > 0$ on $(-\infty, -3)$ and $(-1, \infty)$
$f(x) < 0$ on $(-3, -1)$.

27. $f(x) = \dfrac{x^2 - 1}{x^2 - 4} = \dfrac{(x - 1)(x + 1)}{(x - 2)(x + 2)}$
$f = 0$ at $x = \pm 1$.
f is not defined at $x = \pm 2$.
$f(x) > 0$ on $(-\infty, -2)$, $(-1, 1)$, and $(2, \infty)$.
$f(x) < 0$ on $(-2, -1)$ and $(1, 2)$.

28. $f(x) = \dfrac{x^2 + x - 2}{x^3} = \dfrac{(x + 2)(x - 1)}{x^3}$
$f(x) > 0$ on $(-2, 0)$ and $(1, \infty)$
$f(x) < 0$ on $(-\infty, -2)$ and $(0, 1)$.

29. $f(x) = x^3 + x - 1$, $f(0) = -1$, $f(1) = 1$.
Since f is continuous and changes sign between 0 and 1, it must be zero at some point between 0 and 1 by IVT.

30. $f(x) = x^3 - 15x + 1$ is continuous everywhere.
$f(-4) = -3$, $f(-3) = 19$, $f(1) = -13$, $f(4) = 5$.
Because of the sign changes f has a zero between -4 and -3, another zero between -3 and 1, and another between 1 and 4.

31. $F(x) = (x - a)^2(x - b)^2 + x$. Without loss of generality, we can assume that $a < b$. Being a polynomial, F is continuous on $[a, b]$. Also $F(a) = a$ and $F(b) = b$. Since $a < \frac{1}{2}(a + b) < b$, the Intermediate-Value Theorem guarantees that there is an x in (a, b) such that $F(x) = (a + b)/2$.

32. Let $g(x) = f(x) - x$. Since $0 \leq f(x) \leq 1$ if $0 \leq x \leq 1$, therefore, $g(0) \geq 0$ and $g(1) \leq 0$. If $g(0) = 0$ let $c = 0$, or if $g(1) = 0$ let $c = 1$. (In either case $f(c) = c$.) Otherwise, $g(0) > 0$ and $g(1) < 0$, and, by IVT, there exists c in $(0, 1)$ such that $g(c) = 0$, i.e., $f(c) = c$.

33. The domain of an even function is symmetric about the y-axis. Since f is continuous on the right at $x = 0$, therefore it must be defined on an interval $[0, h]$ for some $h > 0$. Being even, f must therefore be defined on $[-h, h]$. If $x = -y$, then

$$\lim_{x \to 0-} f(x) = \lim_{y \to 0+} f(-y) = \lim_{y \to 0+} f(y) = f(0).$$

Thus, f is continuous on the left at $x = 0$. Being continuous on both sides, it is therefore continuous.

34. f odd $\Leftrightarrow f(-x) = -f(x)$
f continuous on the right $\Leftrightarrow \lim_{x \to 0+} f(x) = f(0)$
Therefore, letting $t = -x$, we obtain

$$\begin{aligned} \lim_{x \to 0-} f(x) &= \lim_{t \to 0+} f(-t) = \lim_{t \to 0+} -f(t) \\ &= -f(0) = f(-0) = f(0). \end{aligned}$$

Therefore f is continuous at 0 and $f(0) = 0$.

35. max 1.593 at -0.831, min -0.756 at 0.629

36. max 0.133 at $x = 1.437$; min -0.232 at $x = -1.805$

37. max 10.333 at $x = 3$; min 4.762 at $x = 1.260$

38. max 1.510 at $x = 0.465$; min 0 at $x = 0$ and $x = 1$

39. root $x = 0.682$

40. root $x = 0.739$

41. roots $x = -0.637$ and $x = 1.410$

42. roots $x = -0.7244919590$ and $x = 1.220744085$

43. fsolve gives an approximation to the single real root to 10 significant figures; solve gives the three roots (including a complex conjugate pair) in exact form involving the quantity $\left(108 + 12\sqrt{69}\right)^{1/3}$; evalf(solve) gives approximations to the three roots using 10 significant figures for the real and imaginary parts.

Section 1.5 The Formal Definition of Limit (page 92)

1. We require $39.9 \le L \le 40.1$. Thus

$$\begin{aligned} 39.9 &\le 39.6 + 0.025T \le 40.1 \\ 0.3 &\le 0.025T \le 0.5 \\ 12 &\le T \le 20. \end{aligned}$$

The temperature should be kept between 12 °C and 20 °C.

2. Since 1.2% of 8,000 is 96, we require the edge length x of the cube to satisfy $7904 \le x^3 \le 8096$. It is sufficient that $19.920 \le x \le 20.079$. The edge of the cube must be within 0.079 cm of 20 cm.

3. $$\begin{aligned} 3 - 0.02 &\le 2x - 1 \le 3 + 0.02 \\ 3.98 &\le 2x \le 4.02 \\ 1.99 &\le x \le 2.01 \end{aligned}$$

4. $$\begin{aligned} 4 - 0.1 &\le x^2 \le 4 + 0.1 \\ 1.9749 &\le x \le 2.0024 \end{aligned}$$

5. $$\begin{aligned} 1 - 0.1 &\le \sqrt{x} \le 1.1 \\ 0.81 &\le x \le 1.21 \end{aligned}$$

6. $$\begin{aligned} -2 - 0.01 &\le \frac{1}{x} \le -2 + 0.01 \\ -\frac{1}{2.01} &\ge x \ge -\frac{1}{1.99} \\ -0.5025 &\le x \le -0.4975 \end{aligned}$$

7. We need $-0.03 \le (3x + 1) - 7 \le 0.03$, which is equivalent to $-0.01 \le x - 2 \le 0.01$ Thus $\delta = 0.01$ will do.

8. We need $-0.01 \le \sqrt{2x+3} - 3 \le 0.01$. Thus

$$\begin{aligned} 2.99 &\le \sqrt{2x+3} \le 3.01 \\ 8.9401 &\le 2x + 3 \le 9.0601 \\ 2.97005 &\le x \le 3.03005 \\ 3 - 0.02995 &\le x - 3 \le 0.03005. \end{aligned}$$

Here $\delta = 0.02995$ will do.

9. We need $8 - 0.2 \le x^3 \le 8.2$, or $1.9832 \le x \le 2.0165$. Thus, we need $-0.0168 \le x - 2 \le 0.0165$. Here $\delta = 0.0165$ will do.

10. We need $1 - 0.05 \le 1/(x+1) \le 1 + 0.05$, or $1.0526 \ge x + 1 \ge 0.9524$. This will occur if $-0.0476 \le x \le 0.0526$. In this case we can take $\delta = 0.0476$.

11. To be proved: $\lim_{x\to 1}(3x+1) = 4$.
Proof: Let $\epsilon > 0$ be given. Then $|(3x+1) - 4| < \epsilon$ holds if $3|x - 1| < \epsilon$, and so if $|x - 1| < \delta = \epsilon/3$. This confirms the limit.

12. To be proved: $\lim_{x\to 2}(5 - 2x) = 1$.
Proof: Let $\epsilon > 0$ be given. Then $|(5 - 2x) - 1| < \epsilon$ holds if $|2x - 4| < \epsilon$, and so if $|x - 2| < \delta = \epsilon/2$. This confirms the limit.

13. To be proved: $\lim_{x\to 0} x^2 = 0$.
Let $\epsilon > 0$ be given. Then $|x^2 - 0| < \epsilon$ holds if $|x - 0| = |x| < \delta = \sqrt{\epsilon}$.

14. To be proved: $\lim_{x\to 2}\frac{x-2}{1+x^2} = 0$.
Proof: Let $\epsilon > 0$ be given. Then

$$\left|\frac{x-2}{1+x^2} - 0\right| = \frac{|x-2|}{1+x^2} \le |x - 2| < \epsilon$$

provided $|x - 2| < \delta = \epsilon$.

15. To be proved: $\lim_{x\to 1/2}\frac{1-4x^2}{1-2x} = 2$.
Proof: Let $\epsilon > 0$ be given. Then if $x \ne 1/2$ we have

$$\left|\frac{1-4x^2}{1-2x} - 2\right| = |(1+2x) - 2| = |2x - 1| = 2\left|x - \frac{1}{2}\right| < \epsilon$$

provided $|x - \frac{1}{2}| < \delta = \epsilon/2$.

16. To be proved: $\lim_{x\to -2}\frac{x^2+2x}{x+2} = -2$.
Proof: Let $\epsilon > 0$ be given. For $x \ne -2$ we have

$$\left|\frac{x^2+2x}{x+2} - (-2)\right| = |x + 2| < \epsilon$$

provided $|x + 2| < \delta = \epsilon$. This completes the proof.

17. To be proved: $\lim_{x\to 1}\frac{1}{x+1} = \frac{1}{2}$.
Proof: Let $\epsilon > 0$ be given. We have

$$\left|\frac{1}{x+1} - \frac{1}{2}\right| = \left|\frac{1-x}{2(x+1)}\right| = \frac{|x-1|}{2|x+1|}.$$

If $|x - 1| < 1$, then $0 < x < 2$ and $1 < x + 1 < 3$, so that $|x + 1| > 1$. Let $\delta = \min(1, 2\epsilon)$. If $|x - 1| < \delta$, then

$$\left|\frac{1}{x+1} - \frac{1}{2}\right| = \frac{|x-1|}{2|x+1|} < \frac{2\epsilon}{2} = \epsilon.$$

This establishes the required limit.

18. To be proved: $\lim_{x\to-1} \frac{x+1}{x^2-1} = -\frac{1}{2}$.
Proof: Let $\epsilon > 0$ be given. If $x \neq -1$, we have

$$\left|\frac{x+1}{x^2-1} - \left(-\frac{1}{2}\right)\right| = \left|\frac{1}{x-1} - \left(-\frac{1}{2}\right)\right| = \frac{|x+1|}{2|x-1|}.$$

If $|x+1| < 1$, then $-2 < x < 0$, so $-3 < x-1 < -1$ and $|x-1| > 1$. Ler $\delta = \min(1, 2\epsilon)$. If $0 < |x-(-1)| < \delta$ then $|x-1| > 1$ and $|x+1| < 2\epsilon$. Thus

$$\left|\frac{x+1}{x^2-1} - \left(-\frac{1}{2}\right)\right| = \frac{|x+1|}{2|x-1|} < \frac{2\epsilon}{2} = \epsilon.$$

This completes the required proof.

19. To be proved: $\lim_{x\to1} \sqrt{x} = 1$.
Proof: Let $\epsilon > 0$ be given. We have

$$|\sqrt{x}-1| = \left|\frac{x-1}{\sqrt{x}+1}\right| \le |x-1| < \epsilon$$

provided $|x-1| < \delta = \epsilon$. This completes the proof.

20. To be proved: $\lim_{x\to2} x^3 = 8$.
Proof: Let $\epsilon > 0$ be given. We have $|x^3-8| = |x-2||x^2+2x+4|$. If $|x-2| < 1$, then $1 < x < 3$ and $x^2 < 9$. Therefore $|x^2+2x+4| \le 9+2\times3+4 = 19$. If $|x-2| < \delta = \min(1, \epsilon/19)$, then

$$|x^3-8| = |x-2||x^2+2x+4| < \frac{\epsilon}{19}\times 19 = \epsilon.$$

This completes the proof.

21. We say that $\lim_{x\to a-} f(x) = L$ if the following condition holds: for every number $\epsilon > 0$ there exists a number $\delta > 0$, depending on ϵ, such that

$$a-\delta < x < a \quad \text{implies} \quad |f(x)-L| < \epsilon.$$

22. We say that $\lim_{x\to-\infty} f(x) = L$ if the following condition holds: for every number $\epsilon > 0$ there exists a number $R > 0$, depending on ϵ, such that

$$x < -R \quad \text{implies} \quad |f(x)-L| < \epsilon.$$

23. We say that $\lim_{x\to a} f(x) = -\infty$ if the following condition holds: for every number $B > 0$ there exists a number $\delta > 0$, depending on B, such that

$$0 < |x-a| < \delta \quad \text{implies} \quad f(x) < -B.$$

24. We say that $\lim_{x\to\infty} f(x) = \infty$ if the following condition holds: for every number $B > 0$ there exists a number $R > 0$, depending on B, such that

$$x > R \quad \text{implies} \quad f(x) > B.$$

25. We say that $\lim_{x\to a+} f(x) = -\infty$ if the following condition holds: for every number $B > 0$ there exists a number $\delta > 0$, depending on R, such that

$$a < x < a+\delta \quad \text{implies} \quad f(x) < -B.$$

26. We say that $\lim_{x\to a-} f(x) = \infty$ if the following condition holds: for every number $B > 0$ there exists a number $\delta > 0$, depending on B, such that

$$a-\delta < x < a \quad \text{implies} \quad f(x) > B.$$

27. To be proved: $\lim_{x\to1+} \frac{1}{x-1} = \infty$. Proof: Let $B > 0$ be given. We have $\frac{1}{x-1} > B$ if $0 < x-1 < 1/B$, that is, if $1 < x < 1+\delta$, where $\delta = 1/B$. This completes the proof.

28. To be proved: $\lim_{x\to1-} \frac{1}{x-1} = -\infty$. Proof: Let $B > 0$ be given. We have $\frac{1}{x-1} < -B$ if $0 > x-1 > -1/B$, that is, if $1-\delta < x < 1$, where $\delta = 1/B$.. This completes the proof.

29. To be proved: $\lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}} = 0$. Proof: Let $\epsilon > 0$ be given. We have

$$\left|\frac{1}{\sqrt{x^2+1}}\right| = \frac{1}{\sqrt{x^2+1}} < \frac{1}{x} < \epsilon$$

provided $x > R$, where $R = 1/\epsilon$. This completes the proof.

30. To be proved: $\lim_{x\to\infty} \sqrt{x} = \infty$. Proof: Let $B > 0$ be given. We have $\sqrt{x} > B$ if $x > R$ where $R = B^2$. This completes the proof.

31. To be proved: if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = M$, then $L = M$.
Proof: Suppose $L \neq M$. Let $\epsilon = |L-M|/3$. Then $\epsilon > 0$. Since $\lim_{x\to a} f(x) = L$, there exists $\delta_1 > 0$ such that $|f(x)-L| < \epsilon$ if $|x-a| < \delta_1$. Since $\lim_{x\to a} f(x) = M$, there exists $\delta_2 > 0$ such that $|f(x)-M| < \epsilon$ if $|x-a| < \delta_2$. Let $\delta = \min(\delta_1, \delta_2)$. If $|x-a| < \delta$, then

$$\begin{aligned} 3\epsilon = |L-M| &= |(f(x)-M)+(L-f(x)| \\ &\le |f(x)-M| + |f(x)-L| < \epsilon+\epsilon = 2\epsilon. \end{aligned}$$

This implies that $3 < 2$, a contradiction. Thus the original assumption that $L \neq M$ must be incorrect. Therefore $L = M$.

32. To be proved: if $\lim_{x\to a} g(x) = M$, then there exists $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|g(x)| < 1 + |M|$.
Proof: Taking $\epsilon = 1$ in the definition of limit, we obtain a number $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|g(x) - M| < 1$. It follows from this latter inequality that

$$|g(x)| = |(g(x)-M)+M| \le |G(x)-M|+|M| < 1+|M|.$$

33. To be proved: if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then $\lim_{x\to a} f(x)g(x) = LM$.
Proof: Let $\epsilon > 0$ be given. Since $\lim_{x\to a} f(x) = L$, there exists $\delta_1 > 0$ such that $|f(x) - L| < \epsilon/(2(1 + |M|))$ if $0 < |x-a| < \delta_1$. Since $\lim_{x\to a} g(x) = M$, there exists $\delta_2 > 0$ such that $|g(x) - M| < \epsilon/(2(1 + |L|))$ if $0 < |x-a| < \delta_2$. By Exercise 32, there exists $\delta_3 > 0$ such that $|g(x)| < 1 + |M|$ if $0 < |x-a| < \delta_3$. Let $\delta = \min(\delta_1, \delta_2, \delta_3)$. If $|x-a| < \delta$, then

$$\begin{aligned}|f(x)g(x) - LM &= |f(x)g(x) - Lg(x) + Lg(x) - LM|\\ &= |(f(x) - L)g(x) + L(g(x) - M)|\\ &\le |(f(x) - L)g(x)| + |L(g(x) - M)|\\ &= |f(x) - L||g(x)| + |L||g(x) - M|\\ &< \frac{\epsilon}{2(1+|M|)}(1+|M|) + |L|\frac{\epsilon}{2(1+|L|)}\\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Thus $\lim_{x\to a} f(x)g(x) = LM$.

34. To be proved: if $\lim_{x\to a} g(x) = M$ where $M \ne 0$, then there exists $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|g(x)| > |M|/2$.
Proof: By the definition of limit, there exists $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|g(x) - M| < |M|/2$ (since $|M|/2$ is a positive number). This latter inequality implies that

$$|M| = |g(x)+(M-g(x))| \le |g(x)|+|g(x)-M| < |g(x)|+\frac{|M|}{2}.$$

It follows that $|g(x)| > |M| - (|M|/2) = |M|/2$, as required.

35. To be proved: if $\lim_{x\to a} g(x) = M$ where $M \ne 0$, then $\lim_{x\to a} \frac{1}{g(x)} = \frac{1}{M}$.
Proof: Let $\epsilon > 0$ be given. Since $\lim_{x\to a} g(x) = M \ne 0$, there exists $\delta_1 > 0$ such that $|g(x) - M| < \epsilon|M|^2/2$ if $0 < |x-a| < \delta_1$. By Exercise 34, there exists $\delta_2 > 0$ such that $|g(x)| > |M|/2$ if $0 < |x-a| < \delta_3$. Let $\delta = \min(\delta_1, \delta_2)$. If $0 < |x-a| < \delta$, then

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \frac{|M-g(x)|}{|M||g(x)|} < \frac{\epsilon|M|^2}{2}\frac{2}{|M|^2} = \epsilon.$$

This completes the proof.

36. To be proved: if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = M \ne 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$.
Proof: By Exercises 33 and 35 we have

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} f(x) \times \frac{1}{g(x)} = L \times \frac{1}{M} = \frac{L}{M}.$$

37. To be proved: if f is continuous at L and $\lim_{x\to c} g(x) = L$, then $\lim_{x\to c} f(g(x)) = f(L)$.
Proof: Let $\epsilon > 0$ be given. Since f is continuous at L, there exists a number $\gamma > 0$ such that if $|y - L| < \gamma$, then $|f(y) - f(L)| < \epsilon$. Since $\lim_{x\to c} g(x) = L$, there exists $\delta > 0$ such that if $0 < |x-c| < \delta$, then $|g(x) - L| < \gamma$. Taking $y = g(x)$, it follows that if $0 < |x-c| < \delta$, then $|f(g(x)) - f(L)| < \epsilon$, so that $\lim_{x\to c} f(g(x)) = f(L)$.

38. To be proved: if $f(x) \le g(x) \le h(x)$ in an open interval containing $x = a$ (say, for $a - \delta_1 < x < a + \delta_1$, where $\delta_1 > 0$), and if $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then also $\lim_{x\to a} g(x) = L$.
Proof: Let $\epsilon > 0$ be given. Since $\lim_{x\to a} f(x) = L$, there exists $\delta_2 > 0$ such that if $0 < |x-a| < \delta_2$, then $|f(x) - L| < \epsilon/3$. Since $\lim_{x\to a} h(x) = L$, there exists $\delta_3 > 0$ such that if $0 < |x-a| < \delta_3$, then $|h(x)-L| < \epsilon/3$. Let $\delta = \min(\delta_1, \delta_2, \delta_3)$. If $0 < |x-a| < \delta$, then

$$\begin{aligned}|g(x) - L| &= |g(x) - f(x) + f(x) - L|\\ &\le |g(x) - f(x)| + |f(x) - L|\\ &\le |h(x) - f(x)| + |f(x) - L|\\ &= |h(x) - L + L - f(x)| + |f(x) - L|\\ &\le |h(x) - L| + |f(x) - L| + |f(x) - L|\\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

Thus $\lim_{x\to a} g(x) = L$.

Review Exercises 1 (page 93)

1. The average rate of change of x^3 over $[1, 3]$ is

$$\frac{3^3 - 1^3}{3 - 1} = \frac{26}{2} = 13.$$

2. The average rate of change of $1/x$ over $[-2, -1]$ is

$$\frac{(1/(-1)) - (1/(-2))}{-1 - (-2)} = \frac{-1/2}{1} = -\frac{1}{2}.$$

3. The rate of change of x^3 at $x = 2$ is

$$\lim_{h\to 0}\frac{(2+h)^3 - 2^3}{h} = \lim_{h\to 0}\frac{8 + 12h + 6h^2 + h^3 - 8}{h} = \lim_{h\to 0}(12 + 6h + h^2) = 12.$$

4. The rate of change of $1/x$ at $x = -3/2$ is

$$\lim_{h\to 0}\frac{\dfrac{1}{-(3/2)+h} - \left(\dfrac{1}{-3/2}\right)}{h} = \lim_{h\to 0}\frac{\dfrac{2}{2h-3} + \dfrac{2}{3}}{h} = \lim_{h\to 0}\frac{2(3 + 2h - 3)}{3(2h-3)h} = \lim_{h\to 0}\frac{4}{3(2h-3)} = -\frac{4}{9}.$$

5. $\lim_{x\to 1}(x^2 - 4x + 7) = 1 - 4 + 7 = 4$

6. $\lim_{x\to 2}\frac{x^2}{1-x^2} = \frac{2^2}{1-2^2} = -\frac{4}{3}$

7. $\lim_{x\to 1}\frac{x^2}{1-x^2}$ does not exist. The denominator approaches 0 (from both sides) while the numerator does not.

8. $\lim_{x\to 2}\frac{x^2-4}{x^2-5x+6} = \lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)(x-3)} = \lim_{x\to 2}\frac{x+2}{x-3} = -4$

9. $\lim_{x\to 2}\frac{x^2-4}{x^2-4x+4} = \lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)^2} = \lim_{x\to 2}\frac{x+2}{x-2}$ does not exist. The denominator approaches 0 (from both sides) while the numerator does not.

10. $\lim_{x\to 2-}\frac{x^2-4}{x^2-4x+4} = \lim_{x\to 2-}\frac{x+2}{x-2} = -\infty$

11. $\lim_{x\to -2+}\frac{x^2-4}{x^2+4x+4} = \lim_{x\to -2+}\frac{x-2}{x+2} = -\infty$

12. $\lim_{x\to 4}\frac{2-\sqrt{x}}{x-4} = \lim_{x\to 4}\frac{4-x}{(2+\sqrt{x})(x-4)} = -\frac{1}{4}$

13. $$\lim_{x\to 3}\frac{x^2-9}{\sqrt{x}-\sqrt{3}} = \lim_{x\to 3}\frac{(x-3)(x+3)(\sqrt{x}+\sqrt{3})}{x-3} = \lim_{x\to 3}(x+3)(\sqrt{x}+\sqrt{3}) = 12\sqrt{3}$$

14. $$\lim_{h\to 0}\frac{h}{\sqrt{x+3h}-\sqrt{x}} = \lim_{h\to 0}\frac{h(\sqrt{x+3h}+\sqrt{x})}{(x+3h)-x} = \lim_{h\to 0}\frac{\sqrt{x+3h}+\sqrt{x}}{3} = \frac{2\sqrt{x}}{3}$$

15. $\lim_{x\to 0+}\sqrt{x-x^2} = 0$

16. $\lim_{x\to 0}\sqrt{x-x^2}$ does not exist because $\sqrt{x-x^2}$ is not defined for $x < 0$.

17. $\lim_{x\to 1}\sqrt{x-x^2}$ does not exist because $\sqrt{x-x^2}$ is not defined for $x > 1$.

18. $\lim_{x\to 1-}\sqrt{x-x^2} = 0$

19. $\lim_{x\to\infty}\frac{1-x^2}{3x^2-x-1} = \lim_{x\to\infty}\frac{(1/x^2)-1}{3-(1/x)-(1/x^2)} = -\frac{1}{3}$

20. $\lim_{x\to -\infty}\frac{2x+100}{x^2+3} = \lim_{x\to -\infty}\frac{(2/x)+(100/x^2)}{1+(3/x^2)} = 0$

21. $\lim_{x\to -\infty}\frac{x^3-1}{x^2+4} = \lim_{x\to -\infty}\frac{x-(1/x^2)}{1+(4/x^2)} = -\infty$

22. $\lim_{x\to\infty}\frac{x^4}{x^2-4} = \lim_{x\to\infty}\frac{x^2}{1-(4/x^2)} = \infty$

23. $\lim_{x\to 0+}\frac{1}{\sqrt{x-x^2}} = \infty$

24. $\lim_{x\to 1/2}\frac{1}{\sqrt{x-x^2}} = \frac{1}{\sqrt{1/4}} = 2$

25. $\lim_{x\to\infty}\sin x$ does not exist; $\sin x$ takes the values -1 and 1 in any interval (R,∞), and limits, if they exist, must be unique.

26. $\lim_{x\to\infty}\frac{\cos x}{x} = 0$ by the squeeze theorem, since

$$-\frac{1}{x} \le \frac{\cos x}{x} \le \frac{1}{x} \quad \text{for all } x > 0$$

and $\lim_{x\to\infty}(-1/x) = \lim_{x\to\infty}(1/x) = 0$.

27. $\lim_{x\to 0}x\sin\frac{1}{x} = 0$ by the squeeze theorem, since

$$-|x| \le x\sin\frac{1}{x} \le |x| \quad \text{for all } x \ne 0$$

and $\lim_{x\to 0}(-|x|) = \lim_{x\to 0}|x| = 0$.

28. $\lim_{x\to 0}\sin\frac{1}{x^2}$ does not exist; $\sin(1/x^2)$ takes the values -1 and 1 in any interval $(-\delta,\delta)$, where $\delta > 0$, and limits, if they exist, must be unique.

29. $$\lim_{x\to -\infty}[x + \sqrt{x^2-4x+1}] = \lim_{x\to -\infty}\frac{x^2-(x^2-4x+1)}{x-\sqrt{x^2-4x+1}} = \lim_{x\to -\infty}\frac{4x-1}{x-|x|\sqrt{1-(4/x)+(1/x^2)}} = \lim_{x\to -\infty}\frac{x[4-(1/x)]}{x+x\sqrt{1-(4/x)+(1/x^2)}} = \lim_{x\to -\infty}\frac{4-(1/x)}{1+\sqrt{1-(4/x)+(1/x^2)}} = 2.$$

Note how we have used $|x| = -x$ (in the second last line), because $x \to -\infty$.

30. $\lim_{x\to\infty}[x + \sqrt{x^2-4x+1}] = \infty + \infty = \infty$

31. $f(x) = x^3 - 4x^2 + 1$ is continuous on the whole real line and so is discontinuous nowhere.

32. $f(x) = \dfrac{x}{x+1}$ is continuous everywhere on its domain, which consists of all real numbers except $x = -1$. It is discontinuous nowhere.

33. $f(x) = \begin{cases} x^2 & \text{if } x > 2 \\ x & \text{if } x \le 2 \end{cases}$ is defined everywhere and discontinuous at $x = 2$, where it is, however, left continuous since $\lim_{x\to 2-} f(x) = 2 = f(2)$.

34. $f(x) = \begin{cases} x^2 & \text{if } x > 1 \\ x & \text{if } x \le 1 \end{cases}$ is defined and continuous everywhere, and so discontinuous nowhere. Observe that $\lim_{x\to 1-} f(x) = 1 = \lim_{x\to 1+} f(x)$.

35. $f(x) = H(x-1) = \begin{cases} 1 & \text{if } x \ge 1 \\ 0 & \text{if } x < 1 \end{cases}$ is defined everywhere and discontinuous at $x = 1$ where it is, however, right continuous.

36. $f(x) = H(9-x^2) = \begin{cases} 1 & \text{if } -3 \le x \le 3 \\ 0 & \text{if } x < -3 \text{ or } x > 3 \end{cases}$ is defined everywhere and discontinuous at $x = \pm 3$. It is right continuous at -3 and left continuous at 3.

37. $f(x) = |x| + |x+1|$ is defined and continuous everywhere. It is discontinuous nowhere.

38. $f(x) = \begin{cases} |x|/|x+1| & \text{if } x \ne -1 \\ 1 & \text{if } x = -1 \end{cases}$ is defined everywhere and discontinuous at $x = -1$ where it is neither left nor right continuous since $\lim_{x\to -1} f(x) = \infty$, while $f(-1) = 1$.

Challenging Problems 1 (page 94)

1. Let $0 < a < b$. The average rate of change of x^3 over $[a,b]$ is

$$\frac{b^3 - a^3}{b - a} = b^2 + ab + a^2.$$

The instantaneous rate of change of x^3 at $x = c$ is

$$\lim_{h\to 0} \frac{(c+h)^3 - c^3}{h} = \lim_{h\to 0} \frac{3c^2h + 3ch^2 + h^3}{h} = 3c^2.$$

If $c = \sqrt{(a^2+ab+b^2)/3}$, then $3c^2 = a^2 + ab + b^2$, so the average rate of change over $[a,b]$ is the instantaneous rate of change at $\sqrt{(a^2+ab+b^2)/3}$.
Claim: $\sqrt{(a^2+ab+b^2)/3} > (a+b)/2$.
Proof: Since $a^2 - 2ab + b^2 = (a-b)^2 > 0$, we have

$$\begin{aligned} 4a^2 + 4ab + 4b^2 &> 3a^2 + 6ab + 3b^2 \\ \frac{a^2+ab+b^2}{3} &> \frac{a^2+2ab+b^2}{4} = \left(\frac{a+b}{2}\right)^2 \\ \sqrt{\frac{a^2+ab+b^2}{3}} &> \frac{a+b}{2}. \end{aligned}$$

2. For x near 0 we have $|x-1| = 1-x$ and $|x+1| = x+1$. Thus

$$\lim_{x\to 0} \frac{x}{|x-1| - |x+1|} = \lim_{x\to 0} \frac{x}{(1-x)-(x+1)} = -\frac{1}{2}.$$

3. For x near 3 we have $|5-2x| = 2x-5$, $|x-2| = x-2$, $|x-5| = 5-x$, and $|3x-7| = 3x-7$. Thus

$$\begin{aligned} \lim_{x\to 3} \frac{|5-2x| - |x-2|}{|x-5| - |3x-7|} &= \lim_{x\to 3} \frac{2x-5-(x-2)}{5-x-(3x-7)} \\ &= \lim_{x\to 3} \frac{x-3}{4(3-x)} = -\frac{1}{4}. \end{aligned}$$

4. Let $y = x^{1/6}$. Then we have

$$\begin{aligned} \lim_{x\to 64} \frac{x^{1/3}-4}{x^{1/2}-8} &= \lim_{y\to 2} \frac{y^2-4}{y^3-8} \\ &= \lim_{y\to 2} \frac{(y-2)(y+2)}{(y-2)(y^2+2y+4)} \\ &= \lim_{y\to 2} \frac{y+2}{y^2+2y+4} = \frac{4}{12} = \frac{1}{3}. \end{aligned}$$

5. Use $a - b = \dfrac{a^3 - b^3}{a^2 + ab + b^2}$ to handle the denominator. We have

$$\begin{aligned} &\lim_{x\to 1} \frac{\sqrt{3+x}-2}{\sqrt[3]{7+x}-2} \\ &= \lim_{x\to 1} \frac{3+x-4}{\sqrt{3+x}+2} \times \frac{(7+x)^{2/3} + 2(7+x)^{1/3} + 4}{(7+x)-8} \\ &= \lim_{x\to 1} \frac{(7+x)^{2/3} + 2(7+x)^{1/3} + 4}{\sqrt{3+x}+2} = \frac{4+4+4}{2+2} = 3. \end{aligned}$$

6. $r_+(a) = \dfrac{-1+\sqrt{1+a}}{a}$, $r_-(a) = \dfrac{-1-\sqrt{1+a}}{a}$.

a) $\lim_{a\to 0} r_-(a)$ does not exist. Observe that the right limit is $-\infty$ and the left limit is ∞.

b) From the following table it appears that $\lim_{a\to 0} r_+(a) = 1/2$, the solution of the linear equation $2x - 1 = 0$ which results from setting $a = 0$ in the quadratic equation $ax^2 + 2x - 1 = 0$.

a	$r_+(a)$
1	0.41421
0.1	0.48810
−0.1	0.51317
0.01	0.49876
−0.01	0.50126
0.001	0.49988
−0.001	0.50013

c) $\lim_{a\to 0} r_+(a) = \lim_{a\to 0} \dfrac{\sqrt{1+a}-1}{a}$
$= \lim_{a\to 0} \dfrac{(1+a)-1}{a(\sqrt{1+a}+1)}$
$= \lim_{a\to 0} \dfrac{1}{\sqrt{1+a}+1} = \dfrac{1}{2}.$

7. TRUE or FALSE

a) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a}\big(f(x)+g(x)\big)$ does not exist.
TRUE, because if $\lim_{x\to a}\big(f(x)+g(x)\big)$ were to exist then

$$\lim_{x\to a} g(x) = \lim_{x\to a}\Big(f(x)+g(x)-f(x)\Big) = \lim_{x\to a}\Big(f(x)+g(x)\Big) - \lim_{x\to a} f(x)$$

would also exist.

b) If neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists, then $\lim_{x\to a}\big(f(x)+g(x)\big)$ does not exist.
FALSE. Neither $\lim_{x\to 0} 1/x$ nor $\lim_{x\to 0}(-1/x)$ exist, but $\lim_{x\to 0}\big((1/x)+(-1/x)\big) = \lim_{x\to 0} 0 = 0$ exists.

c) If f is continuous at a, then so is $|f|$.
TRUE. For any two real numbers u and v we have

$$\Big||u|-|v|\Big| \le |u-v|.$$

This follows from

$$|u| = |u-v+v| \le |u-v| + |v|, \quad \text{and}$$
$$|v| = |v-u+u| \le |v-u| + |u| = |u-v| + |u|.$$

Now we have

$$\Big||f(x)|-|f(a)|\Big| \le |f(x)-f(a)|$$

so the left side approaches zero whenever the right side does. This happens when $x\to a$ by the continuity of f at a.

d) If $|f|$ is continuous at a, then so is f.
FALSE. The function $f(x) = \begin{cases} -1 & \text{if } x<0 \\ 1 & \text{if } x\ge 0 \end{cases}$ is discontinuous at $x=0$, but $|f(x)|=1$ everywhere, and so is continuous at $x=0$.

e) If $f(x) < g(x)$ in an interval around a and if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ both exist, then $L<M$.
FALSE. Let $g(x) = \begin{cases} x^2 & \text{if } x\ne 0 \\ 1 & \text{if } x=0 \end{cases}$ and let $f(x) = -g(x)$. Then $f(x) < g(x)$ for all x, but $\lim_{x\to 0} f(x) = 0 = \lim_{x\to 0} g(x)$. (Note: under the given conditions, it is TRUE that $L\le M$, but not necessarily true that $L<M$.)

8. a) To be proved: if f is a continuous function defined on a closed interval $[a,b]$, then the range of f is a closed interval.
Proof: By the Max-Min Theorem there exist numbers u and v in $[a,b]$ such that $f(u)\le f(x)\le f(v)$ for all x in $[a,b]$. By the Intermediate-Value Theorem, $f(x)$ takes on all values between $f(u)$ and $f(v)$ at values of x between u and v, and hence at points of $[a,b]$. Thus the range of f is $[f(u), f(v)]$, a closed interval.

b) If the domain of the continuous function f is an open interval, the range of f can be any interval (open, closed, half open, finite, or infinite).

9. $f(x) = \dfrac{x^2-1}{|x^2-1|} = \begin{cases} -1 & \text{if } -1<x<1 \\ 1 & \text{if } x<-1 \text{ or } x>1 \end{cases}.$
f is continuous wherever it is defined, that is at all points except $x=\pm 1$. f has left and right limits -1 and 1, respectively, at $x=1$, and has left and right limits 1 and -1, respectively, at $x=-1$. It is not, however, discontinuous at any point, since -1 and 1 are not in its domain.

10. $f(x) = \dfrac{1}{x-x^2} = \dfrac{1}{\frac14 - \left(\frac14 - x + x^2\right)} = \dfrac{1}{\frac14 - \left(x-\frac12\right)^2}.$
Observe that $f(x)\ge f(1/2)=4$ for all x in $(0,1)$.

11. Suppose f is continuous on $[0,1]$ and $f(0)=f(1)$.

a) To be proved: $f(a) = f(a+\frac12)$ for some a in $[0,\frac12]$.
Proof: If $f(1/2)=f(0)$ we can take $a=0$ and be done. If not, let

$$g(x) = f(x+\tfrac12) - f(x).$$

Then $g(0)\ne 0$ and

$$g(1/2) = f(1) - f(1/2) = f(0) - f(1/2) = -g(0).$$

Since g is continuous and has opposite signs at $x=0$ and $x=1/2$, the Intermediate-Value Theorem assures us that there exists a between 0 and 1/2 such that $g(a)=0$, that is, $f(a)=f(a+\frac12)$.

b) To be proved: if $n>2$ is an integer, then $f(a) = f(a+\frac1n)$ for some a in $[0, 1-\frac1n]$.
Proof: Let $g(x) = f(x+\frac1n) - f(x)$. Consider the numbers $x=0$, $x=1/n$, $x=2/n$, $\dots$, $x=(n-1)/n$. If $g(x)=0$ for any of these numbers, then we can let a be that number. Otherwise, $g(x)\ne 0$ at any of these numbers. Suppose that the values of g at all these numbers has the same sign (say positive). Then we have

$$f(1) > f(\tfrac{n-1}{n}) > \cdots > f(\tfrac2n) > \tfrac1n > f(0),$$

which is a contradiction, since $f(0)=f(1)$. Therefore there exists j in the set $\{0,1,2,\dots,n-1\}$ such that $g(j/n)$ and $g((j+1)/n)$ have opposite sign. By the Intermediate-Value Theorem, $g(a)=0$ for some a between j/n and $(j+1)/n$, which we had to prove.