Complete Solutions Manual

Calculus for AP®
with CalcChat® and CalcView®

SECOND EDITION

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Limits and Their Properties

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CHAPTER 1
Limits and Their Properties

Section 1.1 A Preview of Calculus

1. Precalculus: $(20 \text{ ft/sec})(15 \text{ sec}) = 300 \text{ ft}$

2. Calculus required: Velocity is not constant.

Distance $\approx (20 \text{ ft/sec})(15 \text{ sec}) = 300 \text{ ft}$

3. Calculus required: Slope of the tangent line at $x = 2$ is the rate of change, and equals about 0.16.

4. Precalculus: rate of change = slope of line = 0.45

5. Precalculus: Area $= \frac{1}{2}bh = \frac{1}{2}(5)(4) = 10$ sq. units

6. Calculus required: Area $= bh$

$$\begin{aligned} &\approx 2(2.5) \\ &= 5 \text{ sq. units} \end{aligned}$$

7. Calculus required:

Area = Area of square + Area of triangle

$$\begin{aligned} &= bh_1 + \tfrac{1}{2}bh_2 \\ &= 4(4) + \tfrac{1}{2}(4)(1) \\ &= 18 \text{ sq. units} \end{aligned}$$

8. Precalculus:

$$\begin{aligned} \text{Radius of circle} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{[-2 - (-1)]^2 + (1 - 0)^2} \\ &= \sqrt{1 + 1} \\ &= \sqrt{2} \end{aligned}$$

$$\begin{aligned} \text{Area of circle} &= \pi r^2 \\ &= \pi\left(\sqrt{2}\right)^2 \\ &= 2\pi \text{ sq. units} \end{aligned}$$

9. Precalculus:

Area = Area of rectangle + Area of triangle

$$\begin{aligned} &= bh_1 + \tfrac{1}{2}bh_2 \\ &= 6(4) + \tfrac{1}{2}(6)(1) \\ &= 27 \text{ sq. units} \end{aligned}$$

10. Calculus required:

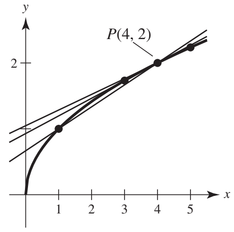
$$\begin{aligned} \text{Area} &= bh \\ &= (1)(1) \\ &= 1 \text{ sq. unit} \end{aligned}$$

11.
$$\begin{aligned} V &= l \cdot w \cdot h \\ &= (4)(2)(3) \\ &= 24 \text{ cubic units} \end{aligned}$$

12.
$$\begin{aligned} V &= \pi r^2 h \\ &= \pi(3)^2(6) \\ &= 54\pi \text{ cubic units} \end{aligned}$$

13. $f(x) = \sqrt{x}$

(a)



(b)
$$\begin{aligned} \text{slope} = m &= \frac{\sqrt{x} - 2}{x - 4} \\ &= \frac{\sqrt{x} - 2}{\left(\sqrt{x} + 2\right)\left(\sqrt{x} - 2\right)} \\ &= \frac{1}{\sqrt{x} + 2}, x \neq 4 \end{aligned}$$

$$x = 1: m = \frac{1}{\sqrt{1} + 2} = \frac{1}{3}$$

$$x = 3: m = \frac{1}{\sqrt{3} + 2} \approx 0.2679$$

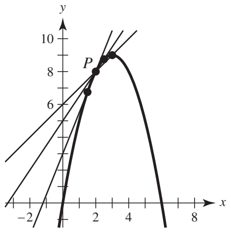
$$x = 5: m = \frac{1}{\sqrt{5} + 2} \approx 0.2361$$

(c) At $P(4, 2)$, the slope is $\dfrac{1}{\sqrt{4} + 2} = \dfrac{1}{4} = 0.25.$

You can improve your approximation of the slope at $x = 4$ by considering x-values very close to 4.

14. $f(x) = 6x - x^2$

(a)



(b) $\text{slope} = m = \dfrac{(6x - x^2) - 8}{x - 2} = \dfrac{(x - 2)(4 - x)}{x - 2}$

$= (4 - x), x \neq 2$

For $x = 3, m = 4 - 3 = 1$

For $x = 2.5, m = 4 - 2.5 = 1.5 = \dfrac{3}{2}$

For $x = 1.5, m = 4 - 1.5 = 2.5 = \dfrac{5}{2}$

(c) At $P(2, 8)$, the slope is 2. You can improve your approximation by considering values of x close to 2.

15. Finding the slope of a curve at a point P is equivalent to finding the slope of the tangent line at P.

16. Answers will vary. Sample answer:
The instantaneous rate of change of an automobile's position is the velocity of the automobile, and can be determined by the speedometer.

17. $\text{Area} \approx 5 + \dfrac{5}{2} + \dfrac{5}{3} + \dfrac{5}{4} \approx 10.417$

$\text{Area} \approx \frac{1}{2}\left(5 + \dfrac{5}{1.5} + \dfrac{5}{2} + \dfrac{5}{2.5} + \dfrac{5}{3} + \dfrac{5}{3.5} + \dfrac{5}{4} + \dfrac{5}{4.5}\right)$

≈ 9.145

You could improve the approximation by using more rectangles.

18. $\text{Area} \approx \dfrac{\pi}{4}\left(\dfrac{\sqrt{2}}{2} + 1 + \dfrac{\sqrt{2}}{2}\right) \approx 1.896$

$\text{Area} \approx \dfrac{\pi}{6}\left(\dfrac{1}{2} + \dfrac{\sqrt{3}}{2} + 1 + \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}\right) \approx 1.954$

You could improve the approximation by using more rectangles.

19. (a) $D_1 = \sqrt{(5 - 1)^2 + (1 - 5)^2} = \sqrt{16 + 16} \approx 5.66$

(b) $D_2 = \sqrt{1 + \left(\frac{5}{2}\right)^2} + \sqrt{1 + \left(\frac{5}{2} - \frac{5}{3}\right)^2} + \sqrt{1 + \left(\frac{5}{3} - \frac{5}{4}\right)^2} + \sqrt{1 + \left(\frac{5}{4} - 1\right)^2} \approx 2.693 + 1.302 + 1.083 + 1.031 \approx 6.11$

(c) The second approximation is more accurate because it uses more line segments to better fit the curve.

20. To obtain a more accurate approximation, increase the number of line segments.

21. Use the sum of the areas of several rectangular regions whose heights correspond to values of $y = f(x)$.

22. Because the graph of f is steeper at $(2, 4)$ than at $(1, 2)$, the slope of the tangent line at $(2, 4)$ is greater than the slope of the tangent line at $(1, 2)$.

23. As point Q approaches point P, the slope of the secant line through P and Q appears to approach 2. So, the answer is B.

24. The area of the region has a base of about 4 units and a height of 5 units, which is greater than an isosceles triangle with an area of $\frac{1}{2}(4)(5) = 10$ square units and less than a rectangle with an area of $(4)(5) = 20$ square units. Because the area of the region appears to be closer to the area of the triangle, the best approximation is 14 square units. So, the answer is B.

Section 1.2 Finding Limits Graphically and Numerically

1.

x	3.9	3.99	3.999	4	4.001	4.01	4.1
$f(x)$	0.3448	0.3344	0.3334	?	0.3332	0.3322	0.3226

$\lim_{x \to 4} \dfrac{x - 4}{x^2 - 5x + 4} \approx 0.3333 \qquad \left(\text{Actual limit is } \dfrac{1}{3}.\right)$

2.

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	0.5132	0.5013	0.5001	?	0.4999	0.4988	0.4881

$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} \approx 0.5000$ $\left(\text{Actual limit is } \frac{1}{2}.\right)$

3.

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	0.9983	0.99998	1.0000	?	1.0000	0.99998	0.9983

$\lim_{x\to 0} \frac{\sin x}{x} \approx 1.0000$ (Actual limit is 1.) (Make sure you use radian mode.)

4.

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	0.0500	0.0050	0.0005	?	–0.0005	–0.0050	–0.0500

$\lim_{x\to 0} \frac{\cos x - 1}{x} \approx 0.0000$ (Actual limit is 0.) (Make sure you use radian mode.)

5.

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	0.9516	0.9950	0.9995	?	1.0005	1.0050	1.0517

$\lim_{x\to 0} \frac{e^x - 1}{x} \approx 1.0000$ (Actual limit is 1.)

6.

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	1.0536	1.0050	1.0005	?	0.9995	0.9950	0.9531

$\lim_{x\to 0} \frac{\ln(x+1)}{x} \approx 1.0000$ (Actual limit is 1.)

7.

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	0.2564	0.2506	0.2501	?	0.2499	0.2494	0.2439

$\lim_{x\to 1} \frac{x-2}{x^2+x-6} \approx 0.2500$ $\left(\text{Actual limit is } \frac{1}{4}.\right)$

8.

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	0.7340	0.6733	0.6673	?	0.6660	0.6600	0.6015

$\lim_{x\to 1} \frac{x^4-1}{x^6-1} \approx 0.6666$ $\left(\text{Actual limit is } \frac{2}{3}.\right)$

9.

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	1.9867	1.9999	2.0000	?	2.0000	1.9999	1.9867

$\lim_{x\to 0} \frac{\sin 2x}{x} \approx 2.0000$ (Actual limit is 2.) (Make sure you use radian mode.)

10.

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	0.4950	0.5000	0.5000	?	0.5000	0.5000	0.4950

$\lim_{x\to 0} \frac{\tan x}{\tan 2x} \approx 0.5000$ $\left(\text{Actual limit is } \frac{1}{2}.\right)$

11.

x	–6.1	–6.01	–6.001	–6	–5.999	–5.99	–5.9
$f(x)$	–0.1248	–0.1250	–0.1250	?	–0.1250	–0.1250	–0.1252

$\lim_{x\to -6} \frac{\sqrt{10-x}-4}{x+6} \approx -0.1250$ $\left(\text{Actual limit is } -\frac{1}{8}.\right)$

12.

x	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$	0.1149	0.115	0.1111	?	0.1111	0.1107	0.1075

$\lim_{x\to 2} \frac{x/(x+1)-2/3}{x-2} \approx 0.1111$ $\left(\text{Actual limit is } \frac{1}{9}.\right)$

13.

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	–2000	-2×10^6	-2×10^9	?	2×10^9	2×10^6	2000

$\lim_{x\to 0} \frac{2}{x^3}$ does not exist; $f(x)$ decreases and increases without bound as $x \to 0$.

14.

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	30	300	3000	?	3000	300	30

$\lim_{x\to 0} \frac{3|x|}{x^2}$ does not exist; $f(x)$ increases without bound as $x \to 0$.

15.

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	3.99982	4	4	?	0	0	0.00018

$\lim_{x\to 0} \frac{4}{1+e^{1/x}}$ does not exist.

16.

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$	−10.6471	−109.5273	−1098.2789	?	1098.9456	110.1940	11.3140

$\lim_{x\to 3} \frac{\ln x}{x-3}$ does not exist; $f(x)$ decreases and increases without bound as $x \to 3$.

17. $\lim_{x\to 3} (4-x) = 1$

18. $\lim_{x\to 0} \sec x = 1$

19. $\lim_{x\to 2} f(x) = \lim_{x\to 2} (4-x) = 2$

20. $\lim_{x\to 1} f(x) = \lim_{x\to 1} (x^2+3) = 4$

21. $\lim_{x\to 2} \frac{|x-2|}{x-2}$ does not exist. The function approaches 1 from the right side of 2, but it approaches −1 from the left side of 2.

22. $\lim_{x\to 0} \dfrac{4}{2 + e^{1/x}}$ does not exist. The function approaches 2 from the left side of 0, but it approaches 0 from the right side of 0.

23. $\lim_{x\to \pi/2} \tan x$ does not exist. The function increases without bound as x approaches $\dfrac{\pi}{2}$ from the left and decreases without bound as x approaches $\dfrac{\pi}{2}$ from the right.

24. $\lim_{x\to 0} \cos(1/x)$ does not exist. The function oscillates between –1 and 1 as x approaches 0.

25. (a) $f(0)$ exists. The graph shows that $f(0) = 3$.

(b) $\lim_{x\to 0} f(x)$ exists. As x approaches 0 from the left and right, $f(x)$ approaches 3. $\lim_{x\to 0} f(x) = 3$.

(c) $f(1)$ exists. The closed circle at (1, 2) indicates that $f(1) = 2$.

(d) $\lim_{x\to 1} f(x)$ does not exist. As x approaches 1 from the left, $f(x)$ approaches 3.5, whereas as x approaches 1 from the right, $f(x)$ approaches 1.

(e) $f(4)$ does not exist. The open circle at $(4, 2)$ indicates that $f(x)$ is not defined at $x = 4$.

(f) $\lim_{x\to 4} f(x)$ exists. As x approaches 4, $f(x)$ approaches 2. $\lim_{x\to 4} f(x) = 2$

26. (a) $f(-2)$ does not exist. The vertical dotted line indicates that f is not defined at –2.

(b) $\lim_{x\to -2} f(x)$ does not exist. As x approaches –2, the values of $f(x)$ do not approach a specific number.

(c) $f(0)$ exists. The closed circle at $(0, 4)$ indicates that $f(0) = 4$.

(d) $\lim_{x\to 0} f(x)$ does not exist. As x approaches 0 from the left, $f(x)$ approaches $\frac{1}{2}$, whereas as x approaches 0 from the right, $f(x)$ approaches 4.

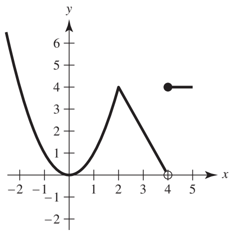
(e) $f(2)$ does not exist. The open circle at $\left(2, \frac{1}{2}\right)$ indicates that $f(2)$ is not defined.

(f) $\lim_{x\to 2} f(x)$ exists. As x approaches 2, $f(x)$ approaches $\frac{1}{2}$. $\lim_{x\to 2} f(x) = \frac{1}{2}$

(g) $f(4)$ exists. The closed circle at $(4, 2)$ indicates that $f(4) = 2$.

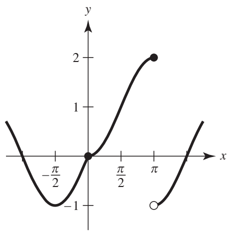
(h) $\lim_{x\to 4} f(x)$ does not exist. As x approaches 4, the values of $f(x)$ do not approach a specific number.

27.



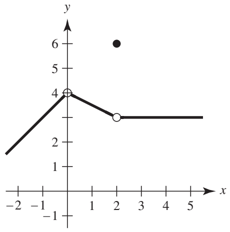
$\lim_{x\to c} f(x)$ exists for all values of $c \neq 4$.

28.

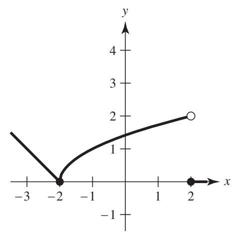


$\lim_{x\to c} f(x)$ exists for all values of $c \neq \pi$.

29. Answers will vary. Sample answer:



30. Answers will vary. Sample answer:



31. You need $|f(x) - 3| = |(x + 1) - 3| = |x - 2| < 0.4$. So, take $\delta = 0.4$. If $0 < |x - 2| < 0.4$, then $|x - 2| = |(x + 1) - 3| = |f(x) - 3| < 0.4$, as desired.

32. You need $\left|f(x) - 1\right| = \left|\frac{1}{x - 1} - 1\right| = \left|\frac{2 - x}{x - 1}\right| < 0.01.$

Let $\delta = \frac{1}{101}$. If $0 < \left|x - 2\right| < \frac{1}{101}$, then

$$-\frac{1}{101} < x - 2 < \frac{1}{101} \Rightarrow 1 - \frac{1}{101} < x - 1 < 1 + \frac{1}{101}$$
$$\Rightarrow \frac{100}{101} < x - 1 < \frac{102}{101}$$
$$\Rightarrow \left|x - 1\right| > \frac{100}{101}$$

and you have

$$\left|f(x) - 1\right| = \left|\frac{1}{x - 1} - 1\right| = \left|\frac{2 - x}{x - 1}\right| < \frac{1/101}{100/101} = \frac{1}{100} = 0.01.$$

33. You need to find δ such that $0 < \left|x - 1\right| < \delta$ implies

$\left|f(x) - 1\right| = \left|\frac{1}{x} - 1\right| < 0.1$. That is,

$$-0.1 < \frac{1}{x} - 1 < 0.1$$
$$1 - 0.1 < \frac{1}{x} < 1 + 0.1$$
$$\frac{9}{10} < \frac{1}{x} < \frac{11}{10}$$
$$\frac{10}{9} > x > \frac{10}{11}$$
$$\frac{10}{9} - 1 > x - 1 > \frac{10}{11} - 1$$
$$\frac{1}{9} > x - 1 > -\frac{1}{11}.$$

So take $\delta = \frac{1}{11}$. Then $0 < \left|x - 1\right| < \delta$ implies

$$-\frac{1}{11} < x - 1 < \frac{1}{11}$$
$$-\frac{1}{11} < x - 1 < \frac{1}{9}.$$

Using the first series of equivalent inequalities, you obtain

$$\left|f(x) - 1\right| = \left|\frac{1}{x} - 1\right| < 0.1.$$

34. You need to find δ such that $0 < \left|x - 2\right| < \delta$ implies $\left|f(x) - 3\right| = \left|x^2 - 1 - 3\right| = \left|x^2 - 4\right| < 0.2$. That is,

$$-0.2 < x^2 - 4 < 0.2$$
$$4 - 0.2 < x^2 < 4 + 0.2$$
$$3.8 < x^2 < 4.2$$
$$\sqrt{3.8} < x < \sqrt{4.2}$$
$$\sqrt{3.8} - 2 < x - 2 < \sqrt{4.2} - 2$$

So take $\delta = \sqrt{4.2} - 2 \approx 0.0494$.

Then $0 < \left|x - 2\right| < \delta$ implies

$$-\left(\sqrt{4.2} - 2\right) < x - 2 < \sqrt{4.2} - 2$$
$$\sqrt{3.8} - 2 < x - 2 < \sqrt{4.2} - 2.$$

Using the first series of equivalent inequalities, you obtain

$$\left|f(x) - 3\right| = \left|x^2 - 4\right| < 0.2.$$

35. $\lim_{x \to 2} (3x + 2) = 3(2) + 2 = 8 = L$

$$\left|(3x + 2) - 8\right| < 0.01$$
$$\left|3x - 6\right| < 0.01$$
$$3\left|x - 2\right| < 0.01$$
$$0 < \left|x - 2\right| < \tfrac{0.01}{3} \approx 0.0033 = \delta$$

So, if $0 < \left|x - 2\right| < \delta = \frac{0.01}{3}$, you have

$$3\left|x - 2\right| < 0.01$$
$$\left|3x - 6\right| < 0.01$$
$$\left|(3x + 2) - 8\right| < 0.01$$
$$\left|f(x) - L\right| < 0.01.$$

36. $\lim_{x \to 6} \left(6 - \frac{x}{3}\right) = 6 - \frac{6}{3} = 4 = L$

$$\left|\left(6 - \frac{x}{3}\right) - 4\right| < 0.01$$
$$\left|2 - \frac{x}{3}\right| < 0.01$$
$$\left|-\tfrac{1}{3}(x - 6)\right| < 0.01$$
$$\left|x - 6\right| < 0.03$$
$$0 < \left|x - 6\right| < 0.03 = \delta$$

So, if $0 < \left|x - 6\right| < \delta = 0.03$, you have

$$\left|-\tfrac{1}{3}(x - 6)\right| < 0.01$$
$$\left|2 - \frac{x}{3}\right| < 0.01$$
$$\left|\left(6 - \frac{x}{3}\right) - 4\right| < 0.01$$
$$\left|f(x) - L\right| < 0.01.$$

37. $\lim_{x\to 2}\left(x^2 - 3\right) = 2^2 - 3 = 1 = L$

$$\left|\left(x^2 - 3\right) - 1\right| < 0.01$$
$$\left|x^2 - 4\right| < 0.01$$
$$\left|(x + 2)(x - 2)\right| < 0.01$$
$$\left|x + 2\right|\left|x - 2\right| < 0.01$$
$$\left|x - 2\right| < \frac{0.01}{\left|x + 2\right|}$$

If you assume $1 < x < 3$, then $\delta \approx 0.01/5 = 0.002$.

So, if $0 < \left|x - 2\right| < \delta \approx 0.002$, you have

$$\left|x - 2\right| < 0.002 = \frac{1}{5}(0.01) < \frac{1}{\left|x + 2\right|}(0.01)$$
$$\left|x + 2\right|\left|x - 2\right| < 0.01$$
$$\left|x^2 - 4\right| < 0.01$$
$$\left|\left(x^2 - 3\right) - 1\right| < 0.01$$
$$\left|f(x) - L\right| < 0.01.$$

38. $\lim_{x\to 4}\left(x^2 + 6\right) = 4^2 + 6 = 22 = L$

$$\left|\left(x^2 + 6\right) - 22\right| < 0.01$$
$$\left|x^2 - 16\right| < 0.01$$
$$\left|(x + 4)(x - 4)\right| < 0.01$$
$$\left|x - 4\right| < \frac{0.01}{\left|x + 4\right|}$$

If you assume $3 < x < 5$, then $\delta = \frac{0.01}{9} \approx 0.00111$.

So, if $0 < \left|x - 4\right| < \delta \approx \frac{0.01}{9}$, you have

$$\left|x - 4\right| < \frac{0.01}{9} < \frac{0.01}{\left|x + 4\right|}$$
$$\left|(x + 4)(x - 4)\right| < 0.01$$
$$\left|x^2 - 16\right| < 0.01$$
$$\left|\left(x^2 + 6\right) - 22\right| < 0.01$$
$$\left|f(x) - L\right| < 0.01.$$

39. $\lim_{x\to 4}(x + 2) = 4 + 2 = 6$

Given $\varepsilon > 0$:

$$\left|(x + 2) - 6\right| < \varepsilon$$
$$\left|x - 4\right| < \varepsilon = \delta$$

So, let $\delta = \varepsilon$. So, if $0 < \left|x - 4\right| < \delta = \varepsilon$, you have

$$\left|x - 4\right| < \varepsilon$$
$$\left|(x + 2) - 6\right| < \varepsilon$$
$$\left|f(x) - L\right| < \varepsilon.$$

40. $\lim_{x\to -2}(4x + 5) = 4(-2) + 5 = -3$

Given $\varepsilon > 0$:

$$\left|(4x + 5) - (-3)\right| < \varepsilon$$
$$\left|4x + 8\right| < \varepsilon$$
$$4\left|x + 2\right| < \varepsilon$$
$$\left|x + 2\right| < \frac{\varepsilon}{4} = \delta$$

So, let $\delta = \frac{\varepsilon}{4}$.

So, if $0 < \left|x + 2\right| < \delta = \frac{\varepsilon}{4}$, you have

$$\left|x + 2\right| < \frac{\varepsilon}{4}$$
$$\left|4x + 8\right| < \varepsilon$$
$$\left|(4x + 5) - (-3)\right| < \varepsilon$$
$$\left|f(x) - L\right| < \varepsilon.$$

41. $\lim_{x\to -4}\left(\frac{1}{2}x - 1\right) = \frac{1}{2}(-4) - 1 = -3$

Given $\varepsilon > 0$:

$$\left|\left(\tfrac{1}{2}x - 1\right) - (-3)\right| < \varepsilon$$
$$\left|\tfrac{1}{2}x + 2\right| < \varepsilon$$
$$\tfrac{1}{2}\left|x - (-4)\right| < \varepsilon$$
$$\left|x - (-4)\right| < 2\varepsilon$$

So, let $\delta = 2\varepsilon$.

So, if $0 < \left|x - (-4)\right| < \delta = 2\varepsilon$, you have

$$\left|x - (-4)\right| < 2\varepsilon$$
$$\left|\tfrac{1}{2}x + 2\right| < \varepsilon$$
$$\left|\left(\tfrac{1}{2}x - 1\right) + 3\right| < \varepsilon$$
$$\left|f(x) - L\right| < \varepsilon.$$

42. $\lim_{x\to 3}\left(\frac{3}{4}x + 1\right) = \frac{3}{4}(3) + 1 = \frac{13}{4}$

Given $\varepsilon > 0$:

$$\left|\left(\frac{3}{4}x + 1\right) - \frac{13}{4}\right| < \varepsilon$$
$$\left|\frac{3}{4}x - \frac{9}{4}\right| < \varepsilon$$
$$\frac{3}{4}|x - 3| < \varepsilon$$
$$|x - 3| < \frac{4}{3}\varepsilon$$

So, let $\delta = \frac{4}{3}\varepsilon$.

So, if $0 < |x - 3| < \delta = \frac{4}{3}\varepsilon$, you have

$$|x - 3| < \frac{4}{3}\varepsilon$$
$$\frac{3}{4}|x - 3| < \varepsilon$$
$$\left|\frac{3}{4}x - \frac{9}{4}\right| < \varepsilon$$
$$\left|\left(\frac{3}{4}x + 1\right) - \frac{13}{4}\right| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

43. $\lim_{x\to 6} 3 = 3$

Given $\varepsilon > 0$:

$$|3 - 3| < \varepsilon$$
$$0 < \varepsilon$$

So, any $\delta > 0$ will work.

So, for any $\delta > 0$, you have

$$|3 - 3| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

44. $\lim_{x\to 2}(-1) = -1$

Given $\varepsilon > 0$:

$$|-1 - (-1)| < \varepsilon$$
$$0 < \varepsilon$$

So, any $\delta > 0$ will work.

So, for any $\delta > 0$, you have

$$|(-1) - (-1)| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

45. $\lim_{x\to 0}\sqrt[3]{x} = 0$

Given $\varepsilon > 0$:

$$\left|\sqrt[3]{x} - 0\right| < \varepsilon$$
$$\left|\sqrt[3]{x}\right| < \varepsilon$$
$$|x| < \varepsilon^3 = \delta$$

So, let $\delta = \varepsilon^3$.

So, for $0|x - 0|\delta = \varepsilon^3$, you have

$$|x| < \varepsilon^3$$
$$\left|\sqrt[3]{x}\right| < \varepsilon$$
$$\left|\sqrt[3]{x} - 0\right| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

46. $\lim_{x\to 4}\sqrt{x} = \sqrt{4} = 2$

Given $\varepsilon > 0$:

$$\left|\sqrt{x} - 2\right| < \varepsilon$$
$$\left|\sqrt{x} - 2\right|\left|\sqrt{x} + 2\right| < \varepsilon\left|\sqrt{x} + 2\right|$$
$$|x - 4| < \varepsilon\left|\sqrt{x} + 2\right|$$

Assuming $1 < x < 9$, you can choose $\delta = 3\varepsilon$. Then,

$$0 < |x - 4| < \delta = 3\varepsilon \Rightarrow |x - 4| < \varepsilon\left|\sqrt{x} + 2\right|$$
$$\Rightarrow \left|\sqrt{x} - 2\right| < \varepsilon.$$

47. $\lim_{x\to -5}|x - 5| = |(-5) - 5| = |-10| = 10$

Given $\varepsilon > 0$:

$$\big||x - 5| - 10\big| < \varepsilon$$
$$|-(x - 5) - 10| < \varepsilon \qquad (x - 5 < 0)$$
$$|-x - 5| < \varepsilon$$
$$|x - (-5)| < \varepsilon$$

So, let $\delta = \varepsilon$.

So for $|x - (-5)| < \delta = \varepsilon$, you have

$$|-(x + 5)| < \varepsilon$$
$$|-(x - 5) - 10| < \varepsilon$$
$$\big||x - 5| - 10\big| < \varepsilon \qquad (\text{because } x - 5 < 0)$$
$$|f(x) - L| < \varepsilon.$$

48. $\lim\limits_{x\to 3} |x - 3| = |3 - 3| = 0$

Given $\varepsilon > 0$:

$$\big||x - 3| - 0\big| < \varepsilon$$
$$|x - 3| < \varepsilon$$

So, let $\delta = \varepsilon$.

So, for $0 < |x - 3| < \delta = \varepsilon$, you have

$$|x - 3| < \varepsilon$$
$$\big||x - 3| - 0\big| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

49. $\lim\limits_{x\to 1} (x^2 + 1) = 1^2 + 1 = 2$

Given $\varepsilon > 0$:

$$|(x^2 + 1) - 2| < \varepsilon$$
$$|x^2 - 1| < \varepsilon$$
$$|(x + 1)(x - 1)| < \varepsilon$$
$$|x - 1| < \frac{\varepsilon}{|x + 1|}$$

If you assume $0 < x < 2$, then $\delta = \varepsilon/3$.

So for $0 < |x - 1| < \delta = \dfrac{\varepsilon}{3}$, you have

$$|x - 1| < \frac{1}{3}\varepsilon < \frac{1}{|x + 1|}\varepsilon$$
$$|x^2 - 1| < \varepsilon$$
$$|(x^2 + 1) - 2| < \varepsilon$$
$$|f(x) - 2| < \varepsilon.$$

50. $\lim\limits_{x\to -4} (x^2 + 4x) = (-4)^2 + 4(-4) = 0$

Given $\varepsilon > 0$:

$$|(x^2 + 4x) - 0| < \varepsilon$$
$$|x(x + 4)| < \varepsilon$$
$$|x + 4| < \frac{\varepsilon}{|x|}$$

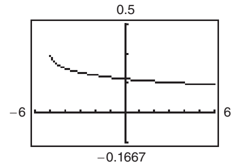
If you assume $-5 < x < -3$, then $\delta = \dfrac{\varepsilon}{5}$.

So for $0 < |x - (-4)| < \delta = \dfrac{\varepsilon}{5}$, you have

$$|x + 4| < \frac{\varepsilon}{5} < \frac{1}{|x|}\varepsilon$$
$$|x(x + 4)| < \varepsilon$$
$$|(x^2 + 4x) - 0| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

51. $f(x) = \dfrac{\sqrt{x + 5} - 3}{x - 4}$

$\lim\limits_{x\to 4} f(x) = \dfrac{1}{6}$

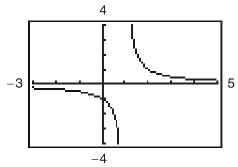


The domain is $[-5, 4) \cup (4, \infty)$.

No, the graphing utility does not show the hole at $\left(4, \frac{1}{6}\right)$. Use the graph to identify asymptotes and intervals, and use the function to identify holes and endpoints.

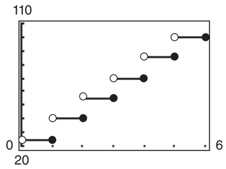
52. $f(x) = \dfrac{x - 3}{x^2 - 4x + 3}$

$\lim\limits_{x\to 3} f(x) = \dfrac{1}{2}$



The domain is $(-\infty, 1) \cup (1, 3) \cup (3, \infty)$. No, the graphing utility does not show the hole at $\left(3, \frac{1}{2}\right)$. Use the graph to identify asymptotes and intervals, and use the function to identify holes and endpoints.

53. (a)



(b)

t	3	3.3	3.4	3.5	3.6	3.7	4
C	55	70	70	?	70	70	70

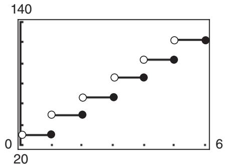
$\lim\limits_{t\to 3.5} C(t) = 70$

(c)

t	2	2.5	2.9	3	3.1	3.5	4
C	40	55	55	?	70	70	70

No, $\lim\limits_{t\to 3} C(t)$ does not exist. As t approaches 3 from the left, $C(t)$ approaches 55, whereas as t approaches 3 from the right, $C(t)$ approaches 70.

54. (a)



(b)

t	3	3.3	3.4	3.5	3.6	3.7	4
C	67	85.5	85.5	?	85.5	85.5	85.5

$\lim_{t\to 3.5} C(t) = 85.5$

(c)

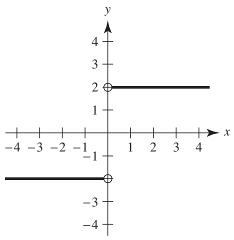
t	2	2.5	2.9	3	3.1	3.5	4
C	48.5	67	67	?	85.5	85.5	85.5

No, $\lim_{t\to 3.5} C(t)$ does not exist. As t approaches 3 from the left, $C(t)$ approaches 67, whereas as t approaches 3 from the right, $C(t)$ approaches 85.5.

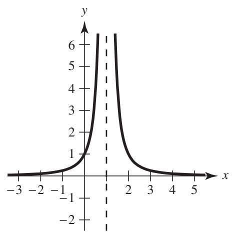
55. As the graph of the function approaches 8 on the horizontal axis, the graph approaches 25 on the vertical axis.

56. In the definition of $\lim_{x\to c} f(x)$, f must be defined on both sides of c, but does not have to be defined at c itself. The value of f at c has no bearing on the limit as x approaches c.

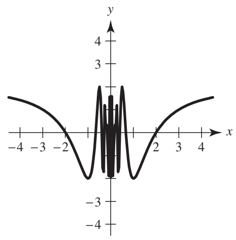
57. (i) The values of f approach different numbers as x approaches c from different sides of c:



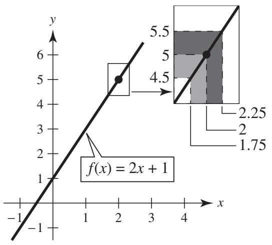
(ii) The values of f increase without bound as x approaches c:



(iii) The values of f oscillate between two fixed numbers as x approaches c:



58.



59. (a) $C = 2\pi r$

$$r = \frac{C}{2\pi} = \frac{6}{2\pi} = \frac{3}{\pi} \text{ cm}$$

(b) When $C = 5.9$: $r = \dfrac{5.9}{2\pi}$ cm

When $C = 6.1$: $r = \dfrac{6.1}{2\pi}$ cm

So, $\dfrac{5.9}{2\pi}$ cm $\le r \le \dfrac{6.1}{2\pi}$ cm.

(c) $\lim_{r\to \frac{3}{\pi}} C = 6$

$\varepsilon = 0.1$

$$\sigma = \frac{0.1}{2\pi} = \frac{1}{20\pi}$$

If r is within $\dfrac{1}{20\pi}$ centimeter of $\dfrac{3}{\pi}$ centimeter, then C will be within 0.1 centimeter of 6 centimeters.

60. (a) $V = \dfrac{4}{3}\pi r^3$

$$r = \sqrt[3]{\frac{3}{4\pi}V} = \sqrt[3]{\frac{3}{4\pi}(2.04)} = \sqrt[3]{\frac{1.53}{\pi}} \text{ in.}$$

(b) When $V = 2.00$: $r = \sqrt[3]{\dfrac{3}{4\pi}(2.00)} = \sqrt[3]{\dfrac{1.5}{\pi}}$ in.

When $V = 2.08$: $r = \sqrt[3]{\dfrac{3}{4\pi}(2.08)} = \sqrt[3]{\dfrac{1.56}{\pi}}$ in.

So, $\sqrt[3]{\dfrac{1.5}{\pi}}$ in. $\le r \le \sqrt[3]{\dfrac{1.56}{\pi}}$ in.

(c) $\lim_{r\to \sqrt[3]{\frac{1.53}{\pi}}} V = 2.04$

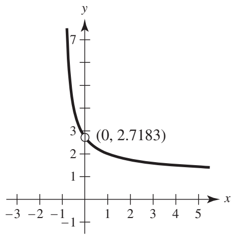
$\varepsilon = 0.04$

$$\sigma = \frac{1}{2}\left(\sqrt[3]{\frac{1.56}{\pi}} - \sqrt[3]{\frac{1.5}{\pi}}\right)$$

If r is within $\dfrac{1}{2}\left(\sqrt[3]{\dfrac{1.56}{\pi}} - \sqrt[3]{\dfrac{1.5}{\pi}}\right)$ inch of $\sqrt[3]{\dfrac{1.53}{\pi}}$ inch, then V will be within 0.04 cubic inch of 2.04 cubic inches.

61. $f(x) = (1 + x)^{1/x}$

$\lim_{x \to 0} (1 + x)^{1/x} = e \approx 2.71828$



x	$f(x)$
–0.1	2.867972
–0.01	2.731999
–0.001	2.719642
–0.0001	2.718418
–0.00001	2.718295
–0.000001	2.718283

x	$f(x)$
0.1	2.593742
0.01	2.704814
0.001	2.716942
0.0001	2.718146
0.00001	2.718268
0.000001	2.718280

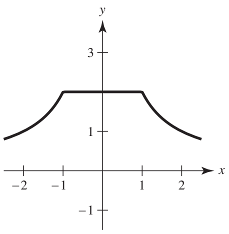
62. $f(x) = \dfrac{|x + 1| - |x - 1|}{x}$

x	–1	–0.5	–0.1	0	0.1	0.5	1.0
$f(x)$	2	2	2	Undef.	2	2	2

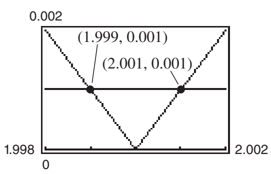
$\lim_{x \to 0} f(x) = 2$

Note that for

$-1 < x < 1, x \neq 0, f(x) = \dfrac{(x + 1) + (x - 1)}{x} = 2.$



63.



Using the zoom and trace feature, $\delta = 0.001$. So $(2 - \delta, 2 + \delta) = (1.999, 2.001)$.

Note: $\dfrac{x^2 - 4}{x - 2} = x + 2$ for $x \neq 2$.

64. (a) $\lim_{x \to c} f(x)$ exists for all $c \neq -3$.

(b) $\lim_{x \to c} f(x)$ exists for all $c \neq -2, 0$.

65. False. The existence or nonexistence of $f(x)$ at $x = c$ has no bearing on the existence of the limit of $f(x)$ as $x \to c$.

66. True

67. False. Let

$$f(x) = \begin{cases} x - 4, & x \neq 2 \\ 0, & x = 2 \end{cases}.$$

So, $f(2) = 0$ and $\lim_{x \to 2} f(x) = \lim_{x \to 2} (x - 4) = 2 \neq 0$.

68. False. Let

$$f(x) = \begin{cases} x - 4, & x \neq 2 \\ 0, & x = 2 \end{cases}.$$

So, $\lim_{x \to 2} f(x) = \lim_{x \to 2} (x - 4) = 2$ and $f(2) = 0 \neq 2$.

69. $f(x) = \sqrt{x}$

$\lim_{x \to 0.25} f(x) = 0.5$ is true.

As x approaches $0.25 = \frac{1}{4}$ from either side,

$f(x) = \sqrt{x}$ approaches $\frac{1}{2} = 0.5$.

70. $f(x) = \sqrt{x}$

$\lim_{x \to 0} f(x) = 0$ is false.

$f(x) = \sqrt{x}$ is not defined on an open interval containing 0 because the domain of f is $x \geq 0$.

71. Using a graphing utility, you can see that

$\lim_{x \to 0} \dfrac{\sin x}{x} = 1$

$\lim_{x \to 0} \dfrac{\sin 2x}{x} = 2,$ etc.

So, $\lim_{x \to 0} \dfrac{\sin nx}{x} = n.$

72. Using a graphing utility, you can see that

$\lim_{x \to 0} \dfrac{\tan x}{x} = 1$

$\lim_{x \to 0} \dfrac{\tan 2x}{x} = 2,$ etc.

So, $\lim_{x \to 0} \dfrac{\tan nx}{x} = n.$

73. If $\lim_{x \to c} f(x) = L_1$ and $\lim_{x \to c} f(x) = L_2$, then for every $\varepsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that $|x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \varepsilon$ and $|x - c| < \delta_2 \Rightarrow |f(x) - L_2| < \varepsilon$. Let δ equal the smaller of δ_1 and δ_2. Then for $|x - c| < \delta$, you have $|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \le |L_1 - f(x)| + |f(x) - L_2| < \varepsilon + \varepsilon$. Therefore, $|L_1 - L_2| < 2\varepsilon$. Because $\varepsilon > 0$ is arbitrary, it follows that $L_1 = L_2$.

74. $f(x) = mx + b, m \ne 0$. Let $\varepsilon > 0$ be given. Take

$$\delta = \frac{\varepsilon}{|m|}.$$

If $0 < |x - c| < \delta = \dfrac{\varepsilon}{|m|}$, then

$$|m||x - c| < \varepsilon$$
$$|mx - mc| < \varepsilon$$
$$|(mx + b) - (mc + b)| < \varepsilon$$

which shows that $\lim_{x \to c} (mx + b) = mc + b$.

75. $\lim_{x \to c}[f(x) - L] = 0$ means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if

$0 < |x - c| < \delta$,

then

$|(f(x) - L) - 0| < \varepsilon$.

This means the same as $|f(x) - L| < \varepsilon$ when

$0 < |x - c| < \delta$.

So, $\lim_{x \to c} f(x) = L$.

76. (a) $(3x + 1)(3x - 1)x^2 + 0.01 = (9x^2 - 1)x^2 + \dfrac{1}{100}$

$$= 9x^4 - x^2 + \frac{1}{100}$$
$$= \frac{1}{100}(10x^2 - 1)(90x^2 - 1)$$

So, $(3x + 1)(3x - 1)x^2 + 0.01 > 0$ if

$10x^2 - 1 < 0$ and $90x^2 - 1 < 0$.

Let $(a, b) = \left(-\dfrac{1}{\sqrt{90}}, \dfrac{1}{\sqrt{90}}\right)$.

For all $x \ne 0$ in (a, b), the graph is positive. You can verify this with a graphing utility.

(b) You are given $\lim_{x \to c} g(x) = L > 0$. Let $\varepsilon = \frac{1}{2}L$. There exists $\delta > 0$ such that $0 < |x - c| < \delta$ implies that $|g(x) - L| < \varepsilon = \dfrac{L}{2}$. That is,

$$-\frac{L}{2} < g(x) - L < \frac{L}{2}$$
$$\frac{L}{2} < g(x) < \frac{3L}{2}$$

For x in the interval $(c - \delta, c + \delta)$, $x \ne c$, you have $g(x) > \dfrac{L}{2} > 0$, as desired.

77. $\lim_{x \to \pi} \sin x = 0$. As x approaches π from the left and right, the graph of $h(x)$ approaches 0. So, the answer is B.

78. The function $f(x) = \dfrac{10}{x^4}$ increases without bound as x approaches 0 from the left and as x approaches 0 from the right. So, the limit is nonexistent, which is answer D.

79. As x approaches 0 from the left and right, the function approaches 2. So, $\lim_{x \to 0} f(x) = 2$, which is answer B.

80. I: As x approaches 3 from the left, $\sqrt{x - 3}$ is not defined. So, $\lim_{x \to 3} \sqrt{x - 3}$ does not exist. Thus, $\lim_{x \to 3} \sqrt{x - 3} = 0$ is not a true statement.

II: As x approaches 3 from the left and right, $6 - 2x$ approaches 0. So, $\lim_{x \to 3} (6 - 2x) = 0$ is a true statement.

III: As x approaches 3 from the left, $f(x) = 6 - 2x$ approaches 0. As x approaches 3 from the right, $f(x) = \sqrt{x - 3}$ approaches 0. So, $\lim_{x \to 3} f(x) = 0$ is a true statement.

Because II and III are true statements, the answer is C.

Section 1.3 Evaluating Limits Analytically

1. $\lim_{x \to 3} 6 = 6$

2. $\lim_{x \to -2} 4 = 4$

3. $\lim_{x \to 4} x = 4$

4. $\lim_{x \to -6} x = -6$

5. $\lim_{x\to 7} x^2 = 7^2 = 49$

6. $\lim_{x\to -2} x^4 = (-2)^4 = 16$

7. (a) $\lim_{x\to 3}\left[3f(x)\right] = 3\left(\lim_{x\to 3} f(x)\right) = 3(2) = 6$

(b) $\lim_{x\to 3}\left[f(x) + g(x)\right] = \lim_{x\to 3} f(x) + \lim_{x\to 3} g(x) = 2 + 6 = 8$

(c) $\lim_{x\to 3}\left[f(x)g(x)\right] = \left[\lim_{x\to 3} f(x)\right]\left[\lim_{x\to 3} g(x)\right] = (2)(6) = 12$

(d) $\lim_{x\to 3}\left[\dfrac{g(x)}{f(x)}\right] = \dfrac{\lim_{x\to 3} g(x)}{\lim_{x\to 3} f(x)} = \dfrac{6}{2} = 3$

8. (a) $\lim_{x\to 1}\left[5g(x)\right] = 5\left(\lim_{x\to 1} g(x)\right) = 5(-1) = -5$

(b) $\lim_{x\to 1}\left[g(x) - f(x)\right] = \lim_{x\to 1} g(x) - \lim_{x\to 1} f(x) = -1 - 2 = -3$

(c) $\lim_{x\to 1}\left[g(x)f(x)\right] = \left[\lim_{x\to 1} g(x)\right]\left[\lim_{x\to 1} f(x)\right] = (-1)(2) = -2$

(d) $\lim_{x\to 1}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to 1} f(x)}{\lim_{x\to 1} g(x)} = \dfrac{2}{-1} = -2$

9. (a) $\lim_{x\to c}\left[f(x)\right]^2 = \left[\lim_{x\to c} f(x)\right]^2 = (16)^2 = 256$

(b) $\lim_{x\to c}\sqrt{f(x)} = \sqrt{\lim_{x\to c} f(x)} = \sqrt{16} = 4$

(c) $\lim_{x\to c}\left[3f(x)\right] = 3\left[\lim_{x\to c} f(x)\right] = 3(16) = 48$

(d) $\lim_{x\to c}\left[f(x)\right]^{3/2} = \left[\lim_{x\to c} f(x)\right]^{3/2} = (16)^{3/2} = 64$

10. (a) $\lim_{x\to c}\sqrt[3]{f(x)} = \sqrt[3]{\lim_{x\to c} f(x)} = \sqrt[3]{27} = 3$

(b) $\lim_{x\to c}\dfrac{f(x)}{18} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} 18} = \dfrac{27}{18} = \dfrac{3}{2}$

(c) $\lim_{x\to c}\left[f(x)\right]^2 = \left[\lim_{x\to c} f(x)\right]^2 = (27)^2 = 729$

(d) $\lim_{x\to c}\left[f(x)\right]^{2/3} = \left[\lim_{x\to c} f(x)\right]^{2/3} = (27)^{2/3} = 9$

11. $\lim_{x\to -3}(2x + 5) = 2(-3) + 5 = -1$

12. $\lim_{x\to 0}(3x - 1) = 3(0) - 1 = -1$

13. $\lim_{x\to -3}\left(x^2 + 3x\right) = (-3)^2 + 3(-3) = 9 - 9 = 0$

14. $\lim_{x\to 2}\left(-x^3 + 1\right) = (-2)^3 + 1 = -8 + 1 = -7$

15. $\lim_{x\to -3}\left(2x^2 + 4x + 1\right) = 2(-3)^2 + 4(-3) + 1 = 18 - 12 + 1 = 7$

16. $\lim_{x\to 1}\left(2x^3 - 6x + 5\right) = 2(1)^3 - 6(1) + 5 = 2 - 6 + 5 = 1$

17. $\lim_{x\to 13}\sqrt{x + 1} = \sqrt{13 + 1} = \sqrt{14} \approx 3.742$

18. $\lim_{x\to 2}\sqrt[3]{12x + 3} = \sqrt[3]{12(2) + 3} = \sqrt[3]{24 + 3} = \sqrt[3]{27} = 3$

19. $\lim_{x\to -4}(1 - x)^3 = \left[1 - (-4)\right]^3 = 5^3 = 125$

20. $\lim_{x\to 0}(3x - 2)^4 = [3(0) - 2]^4 = (-2)^4 = 16$

21. $\lim_{x\to 2}\dfrac{3}{2x + 1} = \dfrac{3}{2(2) + 1} = \dfrac{3}{5}$

22. $\lim_{x\to -5}\dfrac{5}{x + 3} = \dfrac{5}{-5 + 3} = -\dfrac{5}{2}$

23. $\lim_{x\to 1}\dfrac{x}{x^2 + 4} = \dfrac{1}{1^2 + 4} = \dfrac{1}{5}$

24. $\lim_{x\to 1}\dfrac{3x + 5}{x + 1} = \dfrac{3(1) + 5}{1 + 1} = \dfrac{3 + 5}{2} = \dfrac{8}{2} = 4$

25. $\lim_{x\to 7}\dfrac{3x}{\sqrt{x + 2}} = \dfrac{3(7)}{\sqrt{7 + 2}} = \dfrac{21}{3} = 7$

26. $\lim_{x\to 2}\dfrac{\sqrt{x + 7}}{x + 2} = \dfrac{\sqrt{2 + 7}}{2 + 2} = \dfrac{3}{4}$

27. (a) $\lim_{x\to 1} f(x) = 5 - 1 = 4$

(b) $\lim_{x\to 4} g(x) = 4^3 = 64$

(c) $\lim_{x\to 1} g\left(f(x)\right) = g\left(f(1)\right) = g(4) = 64$

28. (a) $\lim_{x \to -3} f(x) = (-3) + 7 = 4$

(b) $\lim_{x \to 4} g(x) = 4^2 = 16$

(c) $\lim_{x \to -3} g(f(x)) = g(4) = 16$

29. (a) $\lim_{x \to 1} f(x) = 4 - 1 = 3$

(b) $\lim_{x \to 3} g(x) = \sqrt{3 + 1} = 2$

(c) $\lim_{x \to 1} g(f(x)) = g(3) = 2$

30. (a) $\lim_{x \to 4} f(x) = 2(4^2) - 3(4) + 1 = 21$

(b) $\lim_{x \to 21} g(x) = \sqrt[3]{21 + 6} = 3$

(c) $\lim_{x \to 4} g(f(x)) = g(21) = 3$

31. This was incorrectly handled as the limit of a composite function, $\lim_{x \to 4} g(f(x))$, rather than a product.

$$\begin{aligned} \lim_{x \to 4} g(x)f(x) &= \left[\lim_{x \to 4} g(x)\right]\left[\lim_{x \to 4} f(x)\right] \\ &= (4 + 1)[2(4) - 5] \\ &= (5)(3) \\ &= 15 \end{aligned}$$

32. In evaluating the limit of this composite function, $\lim_{x \to 4} f(x)$ was not evaluated correctly.

$\lim_{x \to 4} f(x) = 2(4) - 5 = 3$

So,

$\lim_{x \to 4} g(f(x)) = g\left(\lim_{x \to 4} f(x)\right) = g(3) = 3 + 1 = 4.$

33. $\lim_{x \to \pi/2} \sin x = \sin \frac{\pi}{2} = 1$

34. $\lim_{x \to \pi} \tan x = \tan \pi = 0$

35. $\lim_{x \to 1} \cos \frac{\pi x}{3} = \cos \frac{\pi}{3} = \frac{1}{2}$

36. $\lim_{x \to 2} \sin \frac{\pi x}{4} = \sin \frac{2\pi}{4} = \sin \frac{\pi}{2} = 1$

37. $\lim_{x \to 0} \sec 2x = \sec 0 = 1$

38. $\lim_{x \to \pi} \cos 3x = \cos 3\pi = -1$

39. $\lim_{x \to 5\pi/6} \sin x = \sin \frac{5\pi}{6} = \frac{1}{2}$

40. $\lim_{x \to 5\pi/3} \cos x = \cos \frac{5\pi}{3} = \frac{1}{2}$

41. $\lim_{x \to 3} \tan \frac{\pi x}{4} = \tan \frac{3\pi}{4} = -1$

42. $\lim_{x \to 7} \sec \frac{\pi x}{6} = \sec \frac{7\pi}{6} = -\frac{2\sqrt{3}}{3}$

43. $\lim_{x \to 0} e^x \cos 2x = e^0 \cos 0 = 1$

44. $\lim_{x \to 0} e^{-x} \sin \pi x = e^0 \sin 0 = 0$

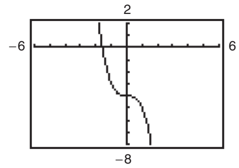
45. $\lim_{x \to 1} (\ln 3x + e^x) = \ln 3 + e$

46. $\lim_{x \to 1} \ln \frac{x}{e^x} = \ln \frac{1}{e} = \ln e^{-1} = -1$

47. $f(x) = \begin{cases} -x^3 - 4, & x \neq -2 \\ -2, & x = -2 \end{cases}$ and $g(x) = -x^3 - 4$

agree except at $x = -2$.

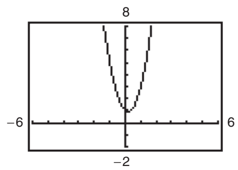
$\lim_{x \to -2} f(x) = \lim_{x \to -2} g(x) = 4$



48. $g(x) = \begin{cases} 3x^2 - x + 1, & x \neq 3 \\ 3, & x = 3 \end{cases}$ and

$h(x) = 3x^2 - x + 1$ agree except at $x = 3$.

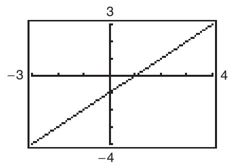
$\lim_{x \to 3} g(x) = \lim_{x \to 3} h(x) = 25$



49. $f(x) = \frac{x^2 - 1}{x + 1} = \frac{(x + 1)(x - 1)}{x + 1}$ and $g(x) = x - 1$

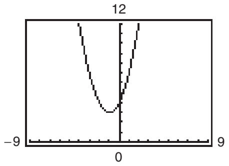
agree except at $x = -1$.

$\lim_{x \to -1} f(x) = \lim_{x \to -1} g(x) = \lim_{x \to -1} (x - 1) = -1 - 1 = -2$



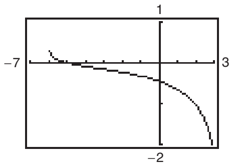
50. $f(x) = \dfrac{x^3 - 8}{x - 2}$ and $g(x) = x^2 + 2x + 4$ agree except at $x = 2$.

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} g(x) = \lim_{x\to 2}\left(x^2 + 2x + 4\right)$$
$$= 2^2 + 2(2) + 4 = 12$$



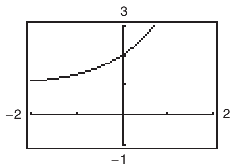
51. $f(x) = \dfrac{(x + 4)\ln(x + 6)}{x^2 - 16}$ and $g(x) = \dfrac{\ln(x + 6)}{x - 4}$ agree except at $x = -4$.

$$\lim_{x\to -4} f(x) = \lim_{x\to -4} g(x) = -\frac{\ln 2}{8} \approx -0.0866$$



52. $f(x) = \dfrac{e^{2x} - 1}{e^x - 1}$ and $g(x) = e^x + 1$ agree except at $x = 0$.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = e^0 + 1 = 2$$



53. $$\lim_{x\to 0}\frac{x}{x^2 - x} = \lim_{x\to 0}\frac{x}{x(x - 1)} = \lim_{x\to 0}\frac{1}{x - 1} = \frac{1}{0 - 1} = -1$$

54. $$\lim_{x\to 0}\frac{x^3 + 9x}{3x} = \lim_{x\to 0}\frac{x\left(x^2 + 9\right)}{3x} = \lim_{x\to 0}\frac{x^2 + 9}{3} = \frac{0^2 + 9}{3} = 3$$

55. $$\lim_{x\to -3}\frac{x^2 - 9}{x + 3} = \lim_{x\to -3}\frac{(x + 3)(x - 3)}{x + 3} = \lim_{x\to -3}(x - 3) = (-3) - 3 = -6$$

56. $$\lim_{x\to 5}\frac{5 - x}{x^2 - 25} = \lim_{x\to 5}\frac{-(x - 5)}{(x - 5)(x + 5)} = \lim_{x\to 5}\frac{-1}{x + 5} = \frac{-1}{5 + 5} = -\frac{1}{10}$$

57. $$\lim_{x\to -3}\frac{x^2 + x - 6}{x^2 - 9} = \lim_{x\to -3}\frac{(x + 3)(x - 2)}{(x + 3)(x - 3)} = \lim_{x\to -3}\frac{x - 2}{x - 3} = \frac{-3 - 2}{-3 - 3} = \frac{-5}{-6} = \frac{5}{6}$$

58. $$\lim_{x\to 2}\frac{x^2 + 2x - 8}{x^2 - x - 2} = \lim_{x\to 2}\frac{(x - 2)(x + 4)}{(x - 2)(x + 1)} = \lim_{x\to 2}\frac{x + 4}{x + 1} = \frac{2 + 4}{2 + 1} = \frac{6}{3} = 2$$

59. $$\lim_{x\to 4}\frac{\sqrt{x + 5} - 3}{x - 4} = \lim_{x\to 4}\frac{\sqrt{x + 5} - 3}{x - 4}\cdot\frac{\sqrt{x + 5} + 3}{\sqrt{x + 5} + 3}$$
$$= \lim_{x\to 4}\frac{(x + 5) - 9}{(x - 4)\left(\sqrt{x + 5} + 3\right)} = \lim_{x\to 4}\frac{1}{\sqrt{x + 5} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{6}$$

60. $$\lim_{x\to 3}\frac{\sqrt{x + 1} - 2}{x - 3} = \lim_{x\to 3}\frac{\sqrt{x + 1} - 2}{x - 3}\cdot\frac{\sqrt{x + 1} + 2}{\sqrt{x + 1} + 2} = \lim_{x\to 3}\frac{x - 3}{(x - 3)\left[\sqrt{x + 1} + 2\right]}$$
$$= \lim_{x\to 3}\frac{1}{\sqrt{x + 1} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4}$$

61. $$\lim_{x\to 0}\frac{\sqrt{x + 5} - \sqrt{5}}{x} = \lim_{x\to 0}\frac{\sqrt{x + 5} - \sqrt{5}}{x}\cdot\frac{\sqrt{x + 5} + \sqrt{5}}{\sqrt{x + 5} + \sqrt{5}}$$
$$= \lim_{x\to 0}\frac{(x + 5) - 5}{x\left(\sqrt{x + 5} + \sqrt{5}\right)} = \lim_{x\to 0}\frac{1}{\sqrt{x + 5} + \sqrt{5}} = \frac{1}{\sqrt{5} + \sqrt{5}} = \frac{1}{2\sqrt{5}} = \frac{\sqrt{5}}{10}$$

62. $\lim_{x \to 0} \dfrac{\sqrt{2 + x} - \sqrt{2}}{x} = \lim_{x \to 0} \dfrac{\sqrt{2 + x} - \sqrt{2}}{x} \cdot \dfrac{\sqrt{2 + x} + \sqrt{2}}{\sqrt{2 + x} + \sqrt{2}}$

$$= \lim_{x \to 0} \frac{2 + x - 2}{\left(\sqrt{2 + x} + \sqrt{2}\right)x} = \lim_{x \to 0} \frac{1}{\sqrt{2 + x} + \sqrt{2}} = \frac{1}{\sqrt{2} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

63. $\lim_{x \to 0} \dfrac{\dfrac{1}{3 + x} - \dfrac{1}{3}}{x} = \lim_{x \to 0} \dfrac{3 - (3 + x)}{(3 + x)3(x)} = \lim_{x \to 0} \dfrac{-x}{(3 + x)(3)(x)} = \lim_{x \to 0} \dfrac{-1}{(3 + x)3} = \dfrac{-1}{(3)3} = -\dfrac{1}{9}$

64. $\lim_{x \to 0} \dfrac{\dfrac{1}{x + 4} - \dfrac{1}{4}}{x} = \lim_{x \to 0} \dfrac{\dfrac{4 - (x + 4)}{4(x + 4)}}{x} = \lim_{x \to 0} \dfrac{-1}{4(x + 4)} = \dfrac{-1}{4(4)} = -\dfrac{1}{16}$

65. $\lim_{\Delta x \to 0} \dfrac{2(x + \Delta x) - 2x}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{2x + 2\Delta x - 2x}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{2\Delta x}{\Delta x} = \lim_{\Delta x \to 0} 2 = 2$

66. $\lim_{\Delta x \to 0} \dfrac{(x + \Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{\Delta x(2x + \Delta x)}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$

67. $\lim_{\Delta x \to 0} \dfrac{(x + \Delta x)^2 - 2(x + \Delta x) + 1 - (x^2 - 2x + 1)}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{x^2 + 2x\Delta x + (\Delta x)^2 - 2x - 2\Delta x + 1 - x^2 + 2x - 1}{\Delta x}$

$$= \lim_{\Delta x \to 0} (2x + \Delta x - 2) = 2x - 2$$

68. $\lim_{\Delta x \to 0} \dfrac{(x + \Delta x)^3 - x^3}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{\Delta x\left(3x^2 + 3x\Delta x + (\Delta x)^2\right)}{\Delta x} = \lim_{\Delta x \to 0} \left(3x^2 + 3x\Delta x + (\Delta x)^2\right) = 3x^2$$

69. $\lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{3(x + \Delta x) - 2 - (3x - 2)}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{3x + 3\Delta x - 2 - 3x + 2}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{3\Delta x}{\Delta x} = 3$

70. $\lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{(x + \Delta x)^2 - 4(x + \Delta x) - (x^2 - 4x)}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{x^2 + 2x\Delta x + \Delta x^2 - 4x - 4\Delta x - x^2 + 4x}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x - 4)}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x - 4) = 2x - 4$$

71. $\lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{\dfrac{1}{x + \Delta x + 3} - \dfrac{1}{x + 3}}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{x + 3 - (x + \Delta x + 3)}{(x + \Delta x + 3)(x + 3)} \cdot \frac{1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-\Delta x}{(x + \Delta x + 3)(x + 3)\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-1}{(x + \Delta x + 3)(x + 3)} = -\frac{1}{(x + 3)^2}$$

72. $f(x) = \dfrac{1}{x^2}$

$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\dfrac{1}{(x+\Delta x)^2}-\dfrac{1}{x^2}}{\Delta x} = \lim_{\Delta x\to 0}\frac{x^2-(x+\Delta x)^2}{x^2(x+\Delta x)^2\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{x^2-\left[x^2+2x\Delta x+(\Delta x)^2\right]}{x^2(x+\Delta x)^2\,\Delta x} = \lim_{\Delta x\to 0}\frac{-2x\Delta x-(\Delta x)^2}{x^2(x+\Delta x)^2\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{-2x-\Delta x}{x^2(x+\Delta x)^2} = \frac{-2x}{x^4} = -\frac{2}{x^3}$$

73. $\displaystyle\lim_{x\to 0}\frac{\sin x}{5x} = \lim_{x\to 0}\left[\left(\frac{\sin x}{x}\right)\left(\frac{1}{5}\right)\right] = (1)\left(\frac{1}{5}\right) = \frac{1}{5}$

74. $\displaystyle\lim_{x\to 0}\frac{3(1-\cos x)}{x} = \lim_{x\to 0}\left[3\left(\frac{(1-\cos x)}{x}\right)\right] = (3)(0) = 0$

75. $\displaystyle\lim_{x\to 0}\frac{\sin x(1-\cos x)}{x^2} = \lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{1-\cos x}{x}\right]$

$= (1)(0) = 0$

76. $\displaystyle\lim_{\theta\to 0}\frac{\cos\theta\tan\theta}{\theta} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

77. $\displaystyle\lim_{x\to 0}\frac{\sin^2 x}{x} = \lim_{x\to 0}\left[\frac{\sin x}{x}\sin x\right] = (1)\sin 0 = 0$

78. $\displaystyle\lim_{x\to 0}\frac{\tan^2 x}{x} = \lim_{x\to 0}\frac{\sin^2 x}{x\cos^2 x} = \lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{\sin x}{\cos^2 x}\right]$

$= (1)(0) = 0$

79. $\displaystyle\lim_{h\to 0}\frac{(1-\cos h)^2}{h} = \lim_{h\to 0}\left[\frac{1-\cos h}{h}(1-\cos h)\right]$

$= (0)(0) = 0$

80. $\displaystyle\lim_{\phi\to\pi}\phi\sec\phi = \pi(-1) = -\pi$

81. $\displaystyle\lim_{x\to 0}\frac{6-6\cos x}{3x} = \lim_{x\to 0}\frac{6(1-\cos x)}{3x}$

$\displaystyle= \lim_{x\to 0}\frac{6}{3}\left(\frac{1-\cos x}{x}\right)$

$\displaystyle= \left(\lim_{x\to 0}2\right)\left(\lim_{x\to 0}\frac{1-\cos x}{x}\right)$

$= (2)(0)$

$= 0$

82. $\displaystyle\lim_{x\to 0}\frac{\cos x-\sin x-1}{2x} = \lim_{x\to 0}\frac{-\sin x}{2x} + \lim_{x\to 0}\frac{\cos x-1}{2x}$

$\displaystyle= -\frac{1}{2}\lim_{x\to 0}\frac{\sin x}{x} - \frac{1}{2}\lim_{x\to 0}\frac{1-\cos x}{x}$

$\displaystyle= -\frac{1}{2}(1) - \frac{1}{2}(0) = -\frac{1}{2}$

83. $\displaystyle\lim_{x\to 0}\frac{1-e^{-x}}{e^x-1} = \lim_{x\to 0}\frac{1-e^{-x}}{e^x-1}\cdot\frac{e^{-x}}{e^{-x}} = \lim_{x\to 0}\frac{(1-e^{-x})e^{-x}}{1-e^{-x}}$

$\displaystyle= \lim_{x\to 0}e^{-x} = 1$

84. $\displaystyle\lim_{x\to 0}\frac{4(e^{2x}-1)}{e^x-1} = \lim_{x\to 0}\frac{4(e^x-1)(e^x+1)}{e^x-1}$

$\displaystyle= \lim_{x\to 0}4(e^x+1) = 4(2) = 8$

85. $\displaystyle\lim_{t\to 0}\frac{\sin 3t}{2t} = \lim_{t\to 0}\left(\frac{\sin 3t}{3t}\right)\left(\frac{3}{2}\right) = (1)\left(\frac{3}{2}\right) = \frac{3}{2}$

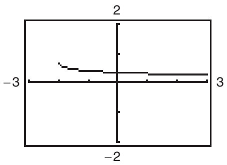
86. $\displaystyle\lim_{x\to 0}\frac{\sin 2x}{\sin 3x} = \lim_{x\to 0}\left[2\left(\frac{\sin 2x}{2x}\right)\left(\frac{1}{3}\right)\left(\frac{3x}{\sin 3x}\right)\right]$

$\displaystyle= 2(1)\left(\frac{1}{3}\right)(1) = \frac{2}{3}$

87. $f(x) = \dfrac{\sqrt{x+2} - \sqrt{2}}{x}$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	0.358	0.354	0.354	?	0.354	0.353	0.349

It appears that the limit is 0.354.



The graph has a hole at $x = 0$.

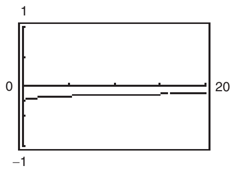
Analytically, $\displaystyle\lim_{x\to 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} = \lim_{x\to 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} \cdot \frac{\sqrt{x+2} + \sqrt{2}}{\sqrt{x+2} + \sqrt{2}}$

$$= \lim_{x\to 0} \frac{x+2-2}{x\left(\sqrt{x+2} + \sqrt{2}\right)} = \lim_{x\to 0} \frac{1}{\sqrt{x+2} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \approx 0.354.$$

88. $f(x) = \dfrac{4 - \sqrt{x}}{x - 16}$

x	15.9	15.99	15.999	16	16.001	16.01	16.1
$f(x)$	–0.1252	–0.125	–0.125	?	–0.125	–0.125	–0.1248

It appears that the limit is –0.125.



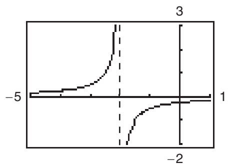
The graph has a hole at $x = 16$.

Analytically, $\displaystyle\lim_{x\to 16} \frac{4 - \sqrt{x}}{x - 16} = \lim_{x\to 16} \frac{\left(4 - \sqrt{x}\right)}{\left(\sqrt{x} + 4\right)\left(\sqrt{x} - 4\right)} = \lim_{x\to 16} \frac{-1}{\sqrt{x} + 4} = -\frac{1}{8}.$

89. $f(x) = \dfrac{\dfrac{1}{2+x} - \dfrac{1}{2}}{x}$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	–0.263	–0.251	–0.250	?	–0.250	–0.249	–0.238

It appears that the limit is –0.250.



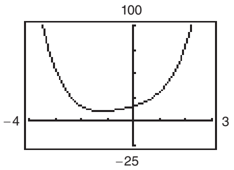
The graph has a hole at $x = 0$.

Analytically, $\displaystyle\lim_{x\to 0} \frac{\dfrac{1}{2+x} - \dfrac{1}{2}}{x} = \lim_{x\to 0} \frac{2 - (2+x)}{2(2+x)} \cdot \frac{1}{x} = \lim_{x\to 0} \frac{-x}{2(2+x)} \cdot \frac{1}{x} = \lim_{x\to 0} \frac{-1}{2(2+x)} = -\frac{1}{4}.$

90. $f(x) = \dfrac{x^5 - 32}{x - 2}$

x	1.9	1.99	1.999	1.9999	2	2.0001	2.001	2.01	2.1
$f(x)$	72.39	79.20	79.92	79.99	?	80.01	80.08	80.80	88.41

It appears that the limit is 80.



The graph has a hole at $x = 2$.

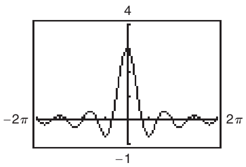
Analytically, $\lim\limits_{x\to 2} \dfrac{x^5 - 32}{x - 2} = \lim\limits_{x\to 2} \dfrac{(x - 2)(x^4 + 2x^3 + 4x^2 + 8x + 16)}{x - 2} = \lim\limits_{x\to 2} (x^4 + 2x^3 + 4x^2 + 8x + 16) = 80.$

(*Hint:* Use long division to factor $x^5 - 32$.)

91. $f(t) = \dfrac{\sin 3t}{t}$

t	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(t)$	2.96	2.9996	3	?	3	2.9996	2.96

It appears that the limit is 3.



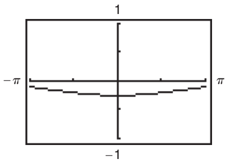
The graph has a hole at $t = 0$.

Analytically, $\lim\limits_{t\to 0} \dfrac{\sin 3t}{t} = \lim\limits_{t\to 0} 3\left(\dfrac{\sin 3t}{3t}\right) = 3(1) = 3.$

92. $f(x) = \dfrac{\cos x - 1}{2x^2}$

x	–1	–0.1	–0.01	0	0.01	0.1	1
$f(x)$	–0.2298	–0.2498	–0.25	?	–0.25	–0.2498	–0.2298

It appears that the limit is –0.25.



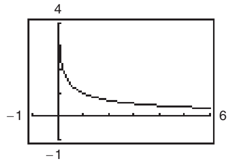
The graph has a hole at $x = 0$.

Analytically, $\dfrac{\cos x - 1}{2x^2} \cdot \dfrac{\cos x + 1}{\cos x + 1} = \dfrac{\cos^2 x - 1}{2x^2(\cos x + 1)} = \dfrac{-\sin^2 x}{2x^2(\cos x + 1)} = \dfrac{\sin^2 x}{x^2} \cdot \dfrac{-1}{2(\cos x + 1)}$

$\lim\limits_{x\to 0}\left[\dfrac{\sin^2 x}{x^2} \cdot \dfrac{-1}{2(\cos x + 1)}\right] = 1\left(\dfrac{-1}{4}\right) = -\dfrac{1}{4} = -0.25.$

93. $f(x) = \dfrac{\ln x}{x - 1}$

x	0.5	0.9	0.99	1	1.01	1.1	1.5
$f(x)$	1.3863	1.0536	1.0050	?	0.9950	0.9531	0.8109

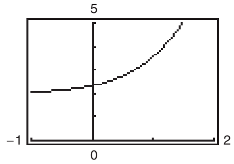


It appears that the limit is 1.

Analytically, $\displaystyle\lim_{x\to 1} \frac{\ln x}{x - 1} = 1.$

94. $f(x) = \dfrac{e^{3x} - 8}{e^{2x} - 4}$

x	0.5	0.6	0.69	ln 2	0.7	0.8	0.9
$f(x)$	2.7450	2.8687	2.9953	0	3.0103	3.1722	3.3565



It appears that the limit is 3.

Analytically, $\displaystyle\lim_{x\to \ln 2} \frac{e^{3x} - 8}{e^{2x} - 4} = \lim_{x\to \ln 2} \frac{(e^x - 2)(e^{2x} + 2e^x + 4)}{(e^x - 2)(e^x + 2)} = \lim_{x\to \ln 2} \frac{e^{2x} + 2e^x + 4}{e^x + 2} = \frac{4 + 4 + 4}{2 + 2} = 3.$

95. $\displaystyle\lim_{x\to 0}(4 - x^2) \le \lim_{x\to 0} f(x) \le \lim_{x\to 0}(4 + x^2)$

$$4 \le \lim_{x\to 0} f(x) \le 4$$

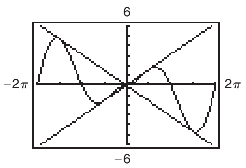
Therefore, $\displaystyle\lim_{x\to 0} f(x) = 4.$

96. $\displaystyle\lim_{x\to a}\left[b - |x - a|\right] \le \lim_{x\to a} f(x) \le \lim_{x\to a}\left[b + |x - a|\right]$

$$b \le \lim_{x\to a} f(x) \le b$$

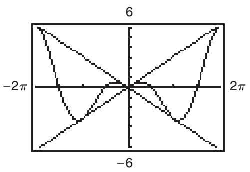
Therefore, $\displaystyle\lim_{x\to a} f(x) = b.$

97. $f(x) = |x|\sin x$



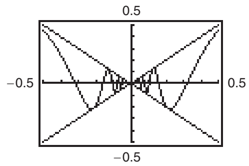
$\displaystyle\lim_{x\to 0} |x|\sin x = 0$

98. $f(x) = |x|\cos x$



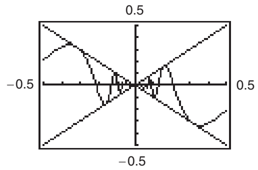
$\displaystyle\lim_{x\to 0} |x|\cos x = 0$

99. $f(x) = x \sin \dfrac{1}{x}$



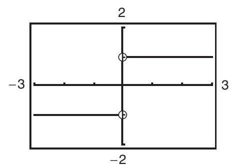
$\displaystyle\lim_{x\to 0}\left(x \sin \frac{1}{x}\right) = 0$

100. $h(x) = x \cos \dfrac{1}{x}$



$\displaystyle\lim_{x\to 0}\left(x \cos \frac{1}{x}\right) = 0$

101. The limit does not exist because the function approaches 1 from the right side of 0 and approaches -1 from the left side of 0.



102. False. $\lim_{x\to\pi} \dfrac{\sin x}{x} = \dfrac{0}{\pi} = 0$

103. True.

104. False. Let $f(x) = \frac{1}{2}x^2$ and $g(x) = x^2$.

Then $f(x) < g(x)$ for all $x \neq 0$. But,

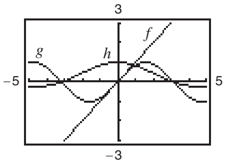
$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0.$

105. (a) Two functions f and g agree at all but one point (on an open interval) if $f(x) = g(x)$ for all x in the interval except for $x = c$, where c is in the interval.

(b) Answers will vary. Sample answer:

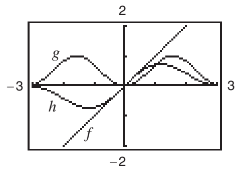
$f(x) = \dfrac{x^2 - 1}{x - 1} = \dfrac{(x + 1)(x - 1)}{x - 1}$ and $g(x) = x + 1$ agree at all points except $x = 1$.

106. $f(x) = x,\ g(x) = \sin x,\ h(x) = \dfrac{\sin x}{x}$



When the x-values are "close to" 0 the magnitude of f is approximately equal to the magnitude of g. So, $|g|/|f| \approx 1$ when x is "close to" 0.

107. $f(x) = x,\ g(x) = \sin^2 x,\ h(x) = \dfrac{\sin^2 x}{x}$



When the x-values are "close to" 0 the magnitude of g is "smaller" than the magnitude of f and the magnitude of g is approaching zero "faster" than the magnitude of f. So, $|g|/|f| \approx 0$ when x is "close to" 0.

108. (a) Use the dividing out technique because the numerator and denominator have a common factor.

$$\lim_{x\to -2} \frac{x^2 + x - 2}{x + 2} = \lim_{x\to -2} \frac{(x + 2)(x - 1)}{x + 2}$$
$$= \lim_{x\to -2} (x - 1) = -2 - 1 = -3$$

(b) Use the rationalizing technique because the numerator involves a radical expression.

$$\lim_{x\to 0} \frac{\sqrt{x + 4} - 2}{x} = \lim_{x\to 0} \frac{\sqrt{x + 4} - 2}{x} - \frac{\sqrt{x + 4} + 2}{\sqrt{x + 4} + 2}$$
$$= \lim_{x\to 0} \frac{(x + 4) - 4}{x\left(\sqrt{x + 4} + 2\right)}$$
$$= \lim_{x\to 0} \frac{1}{\sqrt{x + 4} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4}$$

109. $s(t) = -16t^2 + 500$

$$\lim_{t\to 2} \frac{s(2) - s(t)}{2 - t} = \lim_{t\to 2} \frac{-16(2)^2 + 500 - \left(-16t^2 + 500\right)}{2 - t}$$
$$= \lim_{t\to 2} \frac{436 + 16t^2 - 500}{2 - t}$$
$$= \lim_{t\to 2} \frac{16\left(t^2 - 4\right)}{2 - t}$$
$$= \lim_{t\to 2} \frac{16(t - 2)(t + 2)}{2 - t}$$
$$= \lim_{t\to 2} -16(t + 2) = -64 \text{ ft/sec}$$

The paint can is falling at about 64 feet/second.

110. $s(t) = -16t^2 + 500 = 0$ when $t = \sqrt{\dfrac{500}{16}} = \dfrac{5\sqrt{5}}{2}$ sec. The velocity at time $a = \dfrac{5\sqrt{5}}{2}$ is

$$\lim_{t\to\left(\frac{5\sqrt{5}}{2}\right)} \frac{s\left(\frac{5\sqrt{5}}{2}\right) - s(t)}{\frac{5\sqrt{5}}{2} - t} = \lim_{t\to\left(\frac{5\sqrt{5}}{2}\right)} \frac{0 - \left(-16t^2 + 500\right)}{\frac{5\sqrt{5}}{2} - t}$$

$$= \lim_{t\to\left(\frac{5\sqrt{5}}{2}\right)} \frac{16\left(t^2 - \frac{125}{4}\right)}{\frac{5\sqrt{5}}{2} - t}$$

$$= \lim_{t\to\left(\frac{5\sqrt{5}}{2}\right)} \frac{16\left(t + \frac{5\sqrt{5}}{2}\right)\left(t - \frac{5\sqrt{5}}{2}\right)}{\frac{5\sqrt{5}}{2} - t}$$

$$= \lim_{t\to\frac{5\sqrt{5}}{2}} \left[-16\left(t + \frac{5\sqrt{5}}{2}\right)\right] = -80\sqrt{5} \text{ ft/sec}$$

$$\approx -178.9 \text{ ft/sec.}$$

The velocity of the paint can when it hits the ground is about 178.9 ft/sec.

111. $s(t) = -4.9t^2 + 200$

$$\lim_{t\to 3} \frac{s(3) - s(t)}{3 - t} = \lim_{t\to 3} \frac{-4.9(3)^2 + 200 - \left(-4.9t^2 + 200\right)}{3 - t}$$

$$= \lim_{t\to 3} \frac{4.9\left(t^2 - 9\right)}{3 - t}$$

$$= \lim_{t\to 3} \frac{4.9(t - 3)(t + 3)}{3 - t}$$

$$= \lim_{t\to 3} \left[-4.9(t + 3)\right]$$

$$= -29.4 \text{ m/sec}$$

The object is falling about 29.4 m/sec.

112. $-4.9t^2 + 200 = 0$ when $t = \sqrt{\dfrac{200}{4.9}} = \dfrac{20\sqrt{5}}{7}$ sec. The velocity at time $a = \dfrac{20\sqrt{5}}{7}$ is

$$\lim_{t\to a} \frac{s(a) - s(t)}{a - t} = \lim_{t\to a} \frac{0 - \left[-4.9t^2 + 200\right]}{a - t}$$

$$= \lim_{t\to a} \frac{4.9(t + a)(t - a)}{a - t}$$

$$= \lim_{t\to\frac{20\sqrt{5}}{7}} \left[-4.9\left(t + \frac{20\sqrt{5}}{7}\right)\right] = -28\sqrt{5} \text{ m/sec}$$

$$\approx -62.6 \text{ m/sec.}$$

The velocity of the object when it hits the ground is about 62.6 m/sec.

113. Let $f(x) = 1/x$ and $g(x) = -1 / x$. $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to 0} g(x)$ do not exist. However,

$$\lim_{x\to 0} \left[f(x) + g(x)\right] = \lim_{x\to 0} \left[\frac{1}{x} + \left(-\frac{1}{x}\right)\right] = \lim_{x\to 0} [0] = 0 \text{ and therefore does not exist.}$$

114. Suppose, on the contrary, that $\lim_{x\to c} g(x)$ exists. Then, because $\lim_{x\to c} f(x)$ exists, so would $\lim_{x\to c}\left[f(x) + g(x)\right]$, which is a contradiction. So, $\lim_{x\to c} g(x)$ does not exist.

115. Given $f(x) = b$, show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\left|f(x) - b\right| < \varepsilon$ whenever $\left|x - c\right| < \delta$. Because $\left|f(x) - b\right| = \left|b - b\right| = 0 < \varepsilon$ for every $\varepsilon > 0$, any value of $\delta > 0$ will work.

116. Given $f(x) = x^n$, n is a positive integer, then

$$\begin{aligned}\lim_{x\to c} x^n &= \lim_{x\to c}\left(xx^{n-1}\right)\\ &= \left[\lim_{x\to c} x\right]\left[\lim_{x\to c} x^{n-1}\right] = c\left[\lim_{x\to c}\left(xx^{n-2}\right)\right]\\ &= c\left[\lim_{x\to c} x\right]\left[\lim_{x\to c} x^{n-2}\right] = c(c)\lim_{x\to c}\left(xx^{n-3}\right)\\ &= \cdots = c^n.\end{aligned}$$

117. If $b = 0$, the property is true because both sides are equal to 0. If $b \neq 0$, let $\varepsilon > 0$ be given. Because $\lim_{x\to c} f(x) = L$, there exists $\delta > 0$ such that $\left|f(x) - L\right| < \varepsilon/\left|b\right|$ whenever $0 < \left|x - c\right| < \delta$. So, whenever $0 < \left|x - c\right| < \delta$, we have

$$\left|b\right|\left|f(x) - L\right| < \varepsilon \quad\text{or}\quad \left|bf(x) - bL\right| < \varepsilon$$

which implies that $\lim_{x\to c}\left[bf(x)\right] = bL$.

118. Given $\lim_{x\to c} f(x) = 0$:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\left|f(x) - 0\right| < \varepsilon$ whenever $0 < \left|x - c\right| < \delta$.

Now $\left|f(x) - 0\right| = \left|f(x)\right| = \left|\left|f(x)\right| - 0\right| < \varepsilon$ for $\left|x - c\right| < \delta$. Therefore, $\lim_{x\to c}\left|f(x)\right| = 0$.

119. (a) If $\lim_{x\to c}\left|f(x)\right| = 0$, then $\lim_{x\to c}\left[-\left|f(x)\right|\right] = 0$.

$$-\left|f(x)\right| \le f(x) \le \left|f(x)\right|$$

$$\lim_{x\to c}\left[-\left|f(x)\right|\right] \le \lim_{x\to c} f(x) \le \lim_{x\to c}\left|f(x)\right|$$

$$0 \le \lim_{x\to c} f(x) \le 0$$

Therefore, $\lim_{x\to c} f(x) = 0$.

(b) Given $\lim_{x\to c} f(x) = L$:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\left|f(x) - L\right| < \varepsilon$ whenever $0 < \left|x - c\right| < \delta$.

Because $\left|\left|f(x)\right| - \left|L\right|\right| \le \left|f(x) - L\right| < \varepsilon$ for $\left|x - c\right| < \delta$, then $\lim_{x\to c}\left|f(x)\right| = \left|L\right|$.

120. Let

$$f(x) = \begin{cases} 4, & \text{if } x \ge 0\\ -4, & \text{if } x < 0\end{cases}$$

$\lim_{x\to 0}\left|f(x)\right| = \lim_{x\to 0} 4 = 4.$

$\lim_{x\to 0} f(x)$ does not exist because for $x < 0$, $f(x) = -4$ and for $x \ge 0$, $f(x) = 4$.

121.

$$\begin{aligned}\lim_{x\to 0}\frac{1 - \cos x}{x} &= \lim_{x\to 0}\frac{1 - \cos x}{x}\cdot\frac{1 + \cos x}{1 + \cos x}\\ &= \lim_{x\to 0}\frac{1 - \cos^2 x}{x(1 + \cos x)} = \lim_{x\to 0}\frac{\sin^2 x}{x(1 + \cos x)}\\ &= \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin x}{1 + \cos x}\\ &= \left[\lim_{x\to 0}\frac{\sin x}{x}\right]\left[\lim_{x\to 0}\frac{\sin x}{1 + \cos x}\right]\\ &= (1)(0) = 0\end{aligned}$$

122. $$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational}\\ 1, & \text{if } x \text{ is irrational}\end{cases}$$

$$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational}\\ x, & \text{if } x \text{ is irrational}\end{cases}$$

$\lim_{x\to 0} f(x)$ does not exist.

No matter how "close to" 0 x is, there are still an infinite number of rational and irrational numbers so that $\lim_{x\to 0} f(x)$ does not exist.

$\lim_{x\to 0} g(x) = 0$

When x is "close to" 0, both parts of the function are "close to" 0.

123.

$$\begin{aligned}\lim_{x\to 2}\frac{\sqrt{x + 7} - 3}{x - 2} &= \lim_{x\to 2}\frac{\sqrt{x + 7} - 3}{x - 2}\cdot\frac{\sqrt{x + 7} + 3}{\sqrt{x + 7} + 3}\\ &= \lim_{x\to 2}\frac{(x + 7) - 9}{(x - 2)\left(\sqrt{x + 7} + 3\right)}\\ &= \lim_{x\to 2}\frac{x - 2}{(x - 2)\left(\sqrt{x + 7} + 3\right)}\\ &= \lim_{x\to 2}\frac{1}{\sqrt{x + 7} + 3}\\ &= \frac{1}{\sqrt{2 + 7} + 3}\\ &= \frac{1}{6}\end{aligned}$$

So, the answer is B.

124. Evaluate each limit.

I: Using a graphing utility,

$\lim\limits_{x \to 1} \dfrac{x^3 + 1}{x - 1}$ does not exist.

II: $\lim\limits_{x \to 0} \dfrac{|x|}{x} = \lim\limits_{x \to 0} f(x)$, where $f(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases}$

does not exist because the limits on each side of $x = 0$ do not agree.

III: $\lim\limits_{x \to 2} f(x)$, where $f(x) = \begin{cases} 3, & x \leq 2 \\ 0, & x > 2 \end{cases}$ does not exist

because the limits on each side of $x = 2$ do not agree.

Because the limits of I, II, and III do not exist, the answer is D.

125. $\lim\limits_{x \to 3} \dfrac{x^4 - 81}{x - 3} = \lim\limits_{x \to 3} \dfrac{(x^2 + 9)(x^2 - 9)}{x - 3}$

$= \lim\limits_{x \to 3} \dfrac{(x^2 + 9)(x + 3)(x - 3)}{x - 3}$

$= \lim\limits_{x \to 3} (x^2 + 9)(x + 3)$

$= (3^2 + 9)(3 + 3)$

$= 108$

So, the answer is C.

Section 1.4 Continuity and One-Sided Limits

1. (a) $\lim\limits_{x \to 3^+} f(x) = 1$

(b) $\lim\limits_{x \to 3^-} f(x) = 1$

(c) $\lim\limits_{x \to c} f(x) = 1$

The function is continuous at $x = 3$ and is continuous on $(-\infty, \infty)$.

2. (a) $\lim\limits_{x \to -2^+} f(x) = -2$

(b) $\lim\limits_{x \to -2^-} f(x) = -2$

(c) $\lim\limits_{x \to -2} f(x) = -2$

The function is continuous at $x = -2$ and is continuous on $(-\infty, \infty)$.

3. (a) $\lim\limits_{x \to 3^+} f(x) = 0$

(b) $\lim\limits_{x \to 3^-} f(x) = 0$

(c) $\lim\limits_{x \to 3} f(x) = 0$

The function is not continuous at $x = 3$.

4. (a) $\lim\limits_{x \to -3^+} f(x) = 3$

(b) $\lim\limits_{x \to -3^-} f(x) = 3$

(c) $\lim\limits_{x \to -3} f(x) = 3$

The function is not continuous at $x = -3$.

5. (a) $\lim\limits_{x \to 2^+} f(x) = -3$

(b) $\lim\limits_{x \to 2^-} f(x) = 3$

(c) $\lim\limits_{x \to 2} f(x)$ does not exist

The function is not continuous at $x = 2$.

6. (a) $\lim\limits_{x \to -1^+} f(x) = 0$

(b) $\lim\limits_{x \to -1^-} f(x) = 2$

(c) $\lim\limits_{x \to -1} f(x)$ does not exist.

The function is not continuous at $x = -1$.

7. $\lim\limits_{x \to 8^+} \dfrac{1}{x + 8} = \dfrac{1}{8 + 8} = \dfrac{1}{16}$

8. $\lim\limits_{x \to 3^-} \dfrac{3}{x + 3} = \dfrac{3}{3 + 3} = \dfrac{1}{2}$

9. $\lim\limits_{x \to 4^+} \dfrac{x - 4}{x^2 - 16} = \lim\limits_{x \to 4^+} \dfrac{x - 4}{(x + 4)(x - 4)}$

$= \lim\limits_{x \to 4^+} \dfrac{1}{x + 4}$

$= \dfrac{1}{4 + 4}$

$= \dfrac{1}{8}$

10. $\lim\limits_{x\to 5^+} \dfrac{5 - x}{x^2 - 25} = \lim\limits_{x\to 5^+} \dfrac{-(x - 5)}{(x + 5)(x - 5)}$

$= \lim\limits_{x\to 5^+} \dfrac{-1}{x + 5}$

$= \dfrac{-1}{5 + 5}$

$= -\dfrac{1}{10}$

11. $\lim\limits_{x\to -7^-} \dfrac{x}{\sqrt{x^2 - 49}}$ does not exist because $\dfrac{x}{\sqrt{x^2 - 49}}$ decreases without bound as x approaches -7 from the left.

12. $\lim\limits_{x\to 4^-} \dfrac{\sqrt{x} - 2}{x - 4} = \lim\limits_{x\to 4^-} \dfrac{\sqrt{x} - 2}{x - 4} \cdot \dfrac{\sqrt{x} + 2}{\sqrt{x} + 2}$

$= \lim\limits_{x\to 4^-} \dfrac{x - 4}{(x - 4)(\sqrt{x} + 2)}$

$= \lim\limits_{x\to 4^-} \dfrac{1}{\sqrt{x} + 2} = \dfrac{1}{\sqrt{4} + 2} = \dfrac{1}{4}$

13. $\lim\limits_{x\to 0^-} \dfrac{|x|}{x} = \lim\limits_{x\to 0^-} \dfrac{-x}{x} = -1$

14. $\lim\limits_{x\to 12^+} \dfrac{|x - 12|}{x - 12} = \lim\limits_{x\to 12^+} \dfrac{x - 12}{x - 12} = \lim\limits_{x\to 12^+} 1 = 1$

15. $\lim\limits_{\Delta x\to 0^-} \dfrac{\dfrac{1}{x + \Delta x} - \dfrac{1}{x}}{\Delta x} = \lim\limits_{\Delta x\to 0^-} \dfrac{x - (x + \Delta x)}{x(x + \Delta x)} \cdot \dfrac{1}{\Delta x} = \lim\limits_{\Delta x\to 0^-} \dfrac{-\Delta x}{x(x + \Delta x)} \cdot \dfrac{1}{\Delta x}$

$= \lim\limits_{\Delta x\to 0^-} \dfrac{-1}{x(x + \Delta x)}$

$= \dfrac{-1}{x(x + 0)} = -\dfrac{1}{x^2}$

16. $\lim\limits_{\Delta x\to 0^+} \dfrac{(x + \Delta x)^2 + (x + \Delta x) - (x^2 + x)}{\Delta x} = \lim\limits_{\Delta x\to 0^+} \dfrac{x^2 + 2x(\Delta x) + (\Delta x)^2 + x + \Delta x - x^2 - x}{\Delta x}$

$= \lim\limits_{\Delta x\to 0^+} \dfrac{2x(\Delta x) + (\Delta x)^2 + \Delta x}{\Delta x}$

$= \lim\limits_{\Delta x\to 0^+} (2x + \Delta x + 1)$

$= 2x + 0 + 1 = 2x + 1$

17. $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} \dfrac{x + 2}{2} = \dfrac{5}{2}$

18. $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} (x^2 - 4x + 6) = 9 - 12 + 6 = 3$

$\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} (-x^2 + 4x - 2) = -9 + 12 - 2 = 1$

Because these one-sided limits disagree, $\lim\limits_{x\to 3} f(x)$ does not exist.

19. Limit does not exist. The function decreases without bound as x approaches π from the left and increases without bound as x approaches π from the right.

20. Limit does not exist. The function increases without bound as x approaches $\dfrac{\pi}{2}$ from the left and decreases without bound as x approaches $\dfrac{\pi}{2}$ from the right.

21. $\lim\limits_{x\to 4^-} (7[\![x]\!] - 5) = 7(3) - 5 = 16$

22. $\lim\limits_{x\to 2^+} (4x - [\![x]\!]) = 4(2) - 2 = 6$

23. $\lim\limits_{x\to 2} (3 - [\![-x]\!])$ does not exist because

$\lim\limits_{x\to 2^-} f(x) \ne \lim\limits_{x\to 2^+} f(x)$.

$\lim\limits_{x\to 2^-} (3 - [\![-x]\!]) = 3 - (-2) = 5$

$\lim\limits_{x\to 2^+} (3 - [\![-x]\!]) = 3 - (-3) = 6$

24. $\lim\limits_{x\to 1} \left(1 - \left[\!\!\left[-\dfrac{x}{2}\right]\!\!\right]\right) = 1 - (-1) = 2$

25. $\lim\limits_{x\to 4^+} \ln(x - 4)$ does not exist because $\ln(x - 4)$ decreases without bound as x approaches 4 from the right.

26. $\lim\limits_{x\to 5^-} \ln(5 - x)$ does not exist because $\ln(5 - x)$ decreases without bound as x approaches 5 from the left.

27. $\lim\limits_{x\to 2^-} \ln\left[x^2(4 - x)\right] = \ln\left[2^2(4 - 2)\right] = \ln 8$

28. $\lim\limits_{x\to 10^+} \ln \dfrac{x}{\sqrt{x - 9}} = \ln \dfrac{10}{\sqrt{10 - 9}} = \ln 10$

29. $f(x) = \dfrac{1}{x^2 - 4}$

has discontinuities at $x = -2$ and $x = 2$ because $f(-2)$ and $f(2)$ are not defined.

30. $f(x) = \dfrac{x^2 - 1}{x + 1}$

has a discontinuity at $x = -1$ because $f(-1)$ is not defined.

31. $f(x) = \dfrac{[\![x]\!]}{2} + x$

has discontinuities at each integer k because $\lim\limits_{x\to k^-} f(x) \neq \lim\limits_{x\to k^+} f(x)$.

32. $f(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ 2x - 1, & x > 1 \end{cases}$ has a discontinuity at $x = 1$

because $f(1) = 2 \neq \lim\limits_{x\to 1} f(x) = 1$.

33. $g(x) = \sqrt{49 - x^2}$ is continuous on $[-7, 7]$.

34. $f(t) = 3 - \sqrt{9 - t^2}$ is continuous on $[-3, 3]$.

35. $\lim\limits_{x\to 0^-} f(x) = 3 = \lim\limits_{x\to 0^+} f(x)$. So, f is continuous on $[-1, 4]$.

36. $g(3)$ is not defined. So, g is continuous on $[-1, 3)$.

37. As x approaches -1 from the left, the denominator $x^2 - 1$ is positive and approaches 0. So, $\dfrac{1}{x^2 - 1}$ increases without bound. Thus, $\lim\limits_{x\to -1^-} \dfrac{1}{x^2 - 1}$ does not exist.

38. $g(2)$ is undefined. Because the domain of g is $(-\infty, 2)$, g is continuous on $(-\infty, 2)$.

39. $f(x) = \dfrac{4}{x}$ has a nonremovable discontinuity at $x = 0$ because $\lim\limits_{x\to 0} f(x)$ does not exist.

40. $f(x) = \dfrac{6}{x - 4}$ has a nonremovable discontinuity at $x = 4$ because $\lim\limits_{x\to 4} f(x)$ does not exist.

41. $f(x) = 3x - \cos x$ is continuous for all real x.

42. $f(x) = x^2 - 4x + 4$ is continuous for all real x.

43. $f(x) = \dfrac{1}{4 - x^2} = \dfrac{1}{(2 - x)(2 + x)}$ has nonremovable discontinuities at $x = \pm 2$ because $\lim\limits_{x\to 2} f(x)$ and $\lim\limits_{x\to -2} f(x)$ do not exist.

44. $f(x) = \dfrac{x}{x^2 - x}$ is not continuous at $x = 0, 1$.

Because $\dfrac{x}{x^2 - x} = \dfrac{1}{x - 1}$, $x \neq 0$, $x = 0$ is a removable discontinuity, whereas $x = 1$ is a nonremovable discontinuity.

45. $f(x) = \dfrac{x}{x^2 + 1}$ is continuous for all real x.

46. $f(x) = \dfrac{x - 5}{x^2 - 25} = \dfrac{x - 5}{(x + 5)(x - 5)}$

has a nonremovable discontinuity at $x = -5$ because $\lim\limits_{x\to -5} f(x)$ does not exist, and has a removable discontinuity at $x = 5$ because

$$\lim_{x\to 5} f(x) = \lim_{x\to 5} \frac{1}{x + 5} = \frac{1}{10}.$$

47. $f(x) = \dfrac{x + 2}{x^2 - 3x - 10} = \dfrac{x + 2}{(x + 2)(x - 5)}$

has a nonremovable discontinuity at $x = 5$ because $\lim\limits_{x\to 5} f(x)$ does not exist, and has a removable discontinuity at $x = -2$ because

$$\lim_{x\to -2} f(x) = \lim_{x\to -2} \frac{1}{x - 5} = -\frac{1}{7}.$$

48. $f(x) = \dfrac{x + 2}{x^2 - x - 6} = \dfrac{x + 2}{(x - 3)(x + 2)}$ has a nonremovable discontinuity at $x = 3$ because $\lim\limits_{x\to 3} f(x)$ does not exist, and has a removable discontinuity at $x = -2$ because

$$\lim_{x\to -2} f(x) = \lim_{x\to -2} \frac{1}{x - 3} = -\frac{1}{5}.$$

49. $f(x) = \dfrac{|x + 9|}{x + 9}$ has a nonremovable discontinuity at $x = -9$ because $\lim\limits_{x\to -9} f(x)$ does not exist.

50. $f(x) = \dfrac{|x - 5|}{x - 5}$

has a nonremovable discontinuity at $x = 5$ because $\lim\limits_{x\to 5} f(x)$ does not exist.

51. $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

has a possible discontinuity at $x = 1$.

(1) $f(1) = 1$

(2) $\left.\begin{array}{l} \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} x = 1 \\ \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} x^2 = 1 \end{array}\right\} \lim\limits_{x\to 1} f(x) = 1$

(3) $f(-1) = \lim\limits_{x\to 1} f(x)$

Because f is continuous at $x = 1$, f is continuous for all real x.

52. $f(x) = \begin{cases} -2x + 3, & x < 1 \\ x^2, & x \ge 1 \end{cases}$

has a possible discontinuity at $x = 1$.

(1) $f(1) = 1^2 = 1$

(2) $\left.\begin{array}{l} \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (-2x + 3) = 1 \\ \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} x^2 = 1 \end{array}\right\} \lim\limits_{x\to 1} f(x) = 1$

(3) $f(1) = \lim\limits_{x\to 1} f(x)$

Because f is continuous at $x = 1$, f is continuous for all real x.

53. $f(x) = \begin{cases} \dfrac{x}{2} + 1, & x \le 2 \\ 3 - x, & x > 2 \end{cases}$

has a possible discontinuity at $x = 2$.

(1) $f(2) = \dfrac{2}{2} + 1 = 2$

(2) $\left.\begin{array}{l} \lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} \left(\dfrac{x}{2} + 1\right) = 2 \\ \lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (3 - x) = 1 \end{array}\right\} \lim\limits_{x\to 2} f(x)$ does not exist.

So, f has a nonremovable discontinuity at $x = 2$.

54. $f(x) = \begin{cases} -2x, & x \le 2 \\ x^2 + 1, & x > 2 \end{cases}$

has a possible discontinuity at $x = 2$.

(1) $f(2) = -2(2) = -4$

(2) $\left.\begin{array}{l} \lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} (-2x) = -4 \\ \lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (x^2 + 1) = 5 \end{array}\right\} \lim\limits_{x\to 2} f(x)$ does not exist.

So, f has a nonremovable discontinuity at $x = 2$.

55. $f(x) = \begin{cases} \tan \dfrac{\pi x}{4}, & |x| < 1 \\ x, & |x| \ge 1 \end{cases}$

$= \begin{cases} \tan \dfrac{\pi x}{4}, & -1 < x < 1 \\ x, & x \le -1 \text{ or } x \ge 1 \end{cases}$

has possible discontinuities at $x = -1, x = 1$.

(1) $f(-1) = -1$ $\quad f(1) = 1$

(2) $\lim\limits_{x\to -1} f(x) = -1$ $\quad \lim\limits_{x\to 1} f(x) = 1$

(3) $f(-1) = \lim\limits_{x\to -1} f(x)$ $\quad f(1) = \lim\limits_{x\to 1} f(x)$

Because f is continuous at $x = \pm 1$, f is continuous for all real x.

56. $f(x) = \begin{cases} \csc \dfrac{\pi x}{6}, & |x-3| \le 2 \\ 2, & |x-3| > 2 \end{cases}$

$= \begin{cases} \csc \dfrac{\pi x}{6}, & 1 \le x \le 5 \\ 2, & x < 1 \text{ or } x > 5 \end{cases}$

has possible discontinuities at $x = 1, x = 5$.

(1) $f(1) = \csc \dfrac{\pi}{6} = 2$ $\quad f(5) = \csc \dfrac{5\pi}{6} = 2$

(2) $\lim\limits_{x\to 1} f(x) = 2$ $\quad \lim\limits_{x\to 5} f(x) = 2$

(3) $f(1) = \lim\limits_{x\to 1} f(x)$ $\quad f(5) = \lim\limits_{x\to 5} f(x)$

Because f is continuous at $x = 1$ and $x = 5$, f is continuous for all real x.

57. $f(x) = \begin{cases} \ln(x+1), & x \ge 0 \\ 1 - x^2, & x < 0 \end{cases}$

has a possible discontinuity at $x = 0$.

(1) $f(0) = \ln(0+1) = \ln 1 = 0$

(2) $\left.\begin{aligned} \lim_{x\to 0^-} f(x) &= 1 - 0 = 1 \\ \lim_{x\to 0^+} f(x) &= 0 \end{aligned}\right\} \lim\limits_{x\to 0} f(x)$ does not exist.

So, f has a nonremovable discontinuity at $x = 0$.

58. $f(x) = \begin{cases} 10 - 3e^{5-x}, & x > 5 \\ 10 - \frac{3}{5}x, & x \le 5 \end{cases}$

has a possible discontinuity at $x = 5$.

(1) $f(5) = 7$

(2) $\left.\begin{aligned} \lim_{x\to 5^+} f(x) &= 10 - 3e^{5-5} = 7 \\ \lim_{x\to 5^-} f(x) &= 10 - \tfrac{3}{5}(5) = 7 \end{aligned}\right\} \lim\limits_{x\to 5} f(x) = 7$

(3) $f(5) = \lim\limits_{x\to 5} f(x)$

Because f is continuous at $x = 5$, f is continuous for all real x.

59. $f(x) = \csc x$ has nonremovable discontinuities at integer multiples of π.

60. $f(x) = \tan \dfrac{\pi x}{2}$ has nonremovable discontinuities at each $2k + 1$, where k is an integer.

61. $f(x) = [\![x - 5]\!]$ has nonremovable discontinuities at each integer k because $\lim\limits_{x\to k} f(x)$ does not exist for each integer k.

62. $f(x) = 8 - [\![x]\!]$ has nonremovable discontinuities at each integer k because $\lim\limits_{x\to k} f(x)$ does not exist for each integer k.

63. $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} x^3 = 8$

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} ax^2 = 4a$

Because $4a = 8$, $f(x)$ is continuous when $a = 2$.

64. $\lim\limits_{x\to 0^-} g(x) = \lim\limits_{x\to 0^-} \dfrac{4 \sin x}{x} = 4$

$\lim\limits_{x\to 0^+} g(x) = \lim\limits_{x\to 0^+} (a - 2x) = a$

$f(x)$ is continuous when $a = 4$.

65. Find a and b such that $\lim\limits_{x\to -1^+} (ax + b) = -a + b = 2$ and $\lim\limits_{x\to 3^-} (ax + b) = 3a + b = -2$.

$$\begin{aligned} a - b &= -2 \\ (+)3a + b &= -2 \\ \hline 4a &= -4 \\ a &= -1 \\ b &= 2 + (-1) = 1 \end{aligned}$$

$$f(x) = \begin{cases} 2, & x \le -1 \\ -x + 1, & -1 < x < 3 \\ -2, & x \ge 3 \end{cases}$$

$f(x)$ is continuous when $a = -1$ and $b = 1$.

66. $\lim_{x\to a} g(x) = \lim_{x\to a} \dfrac{x^2 - a^2}{x - a}$

$= \lim_{x\to a} (x + a) = 2a$

Because $2a = 8$, $f(x)$ is continuous when $a = 4$.

67. $f(1) = \arctan(1 - 1) + 2 = 2$

Find a such that $\lim_{x\to 1^-} \left(ae^{x-1} + 3\right) = 2$.

$$ae^{1-1} + 3 = 2$$
$$a + 3 = 2$$
$$a = -1$$

68. $f(4) = 2e^{4a} - 2$

Find a such that $\lim_{x\to 4^+} \ln(x - 3) + x^2 = 2e^{4a} - 2$.

$$\ln(4 - 3) + 4^2 = 2e^{4a} - 2$$
$$16 = 2e^{4a} - 2$$
$$9 = e^{4a}$$
$$\ln 9 = 4a$$
$$a = \frac{\ln 9}{4} = \frac{\ln 3^2}{4} = \frac{\ln 3}{2}$$

69. $f(g(x)) = (x - 1)^2$

Continuous for all real x

70. $f(g(x)) = \dfrac{1}{\sqrt{x - 1}}$

Nonremovable discontinuity at $x = 1$; continuous for all $x > 1$

71. $f(g(x)) = \dfrac{1}{(x^2 + 5) - 6} = \dfrac{1}{x^2 - 1}$

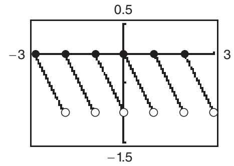
Nonremovable discontinuities at $x = \pm 1$

72. $f(g(x)) = \sin x^2$

Continuous for all real x

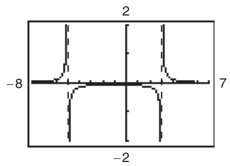
73. $y = [\![x]\!] - x$

Nonremovable discontinuities at each integer



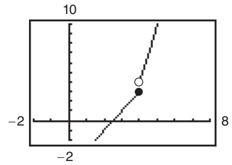
74. $h(x) = \dfrac{1}{x^2 + 2x - 15} = \dfrac{1}{(x + 5)(x - 3)}$

Nonremovable discontinuities at $x = -5$ and $x = 3$



75. $g(x) = \begin{cases} x^2 - 3x, & x > 4 \\ 2x - 5, & x \le 4 \end{cases}$

Nonremovable discontinuity at $x = 4$

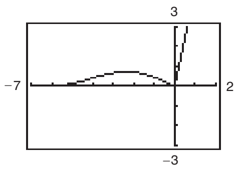


76. $f(x) = \begin{cases} \dfrac{\cos x - 1}{x}, & x < 0 \\ 5x, & x \ge 0 \end{cases}$

$f(0) = 5(0) = 0$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \dfrac{(\cos x - 1)}{x} = 0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (5x) = 0$



Therefore, $\lim_{x\to 0} f(x) = 0 = f(0)$

and f is continuous on the entire real line. ($x = 0$ was the only possible discontinuity.)

77. $f(x) = \dfrac{x^2 - 16}{x - 4}$

f is continuous on $(-\infty, 4) \cup (4, \infty)$.

78. $f(x) = \dfrac{x + 1}{\sqrt{x}}$

f is continuous on $(0, \infty)$.

79. $f(x) = 3 - \sqrt{x}$

f is continuous on $[0, \infty)$.

80. $f(x) = x\sqrt{x + 3}$

f is continuous on $[-3, \infty)$.

81. $f(x) = \sec\dfrac{\pi x}{4}$

f is continuous on
$\ldots, (-6, -2), (-2, 2), (2, 6), (6, 10), \ldots$.

82. $f(x) = \cos\dfrac{1}{x}$

f is continuous on $(-\infty, 0) \cup (0, \infty)$.

83. $f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$

Because

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1} \frac{(x - 1)(x + 1)}{x - 1}$$
$$= \lim_{x\to 1} (x + 1) = 2,$$

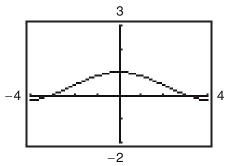
f is continuous on $(-\infty, \infty)$.

84. $f(x) = \begin{cases} 2x - 4, & x \neq 3 \\ 1, & x = 3 \end{cases}$

Because $\lim_{x\to 3} f(x) = \lim_{x\to 3} (2x - 4) = 2 \neq 1$,

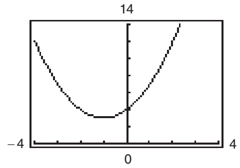
f is continuous on $(-\infty, 3) \cup (3, \infty)$.

85. $f(x) = \dfrac{\sin x}{x}$



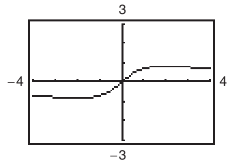
The graph has a hole at $x = 0$. The graph appears to be continuous, but the function is not continuous on $[-4, 4]$. It is not obvious from the graph that the function has a discontinuity at $x = 0$.

86. $f(x) = \dfrac{x^3 - 8}{x - 2}$



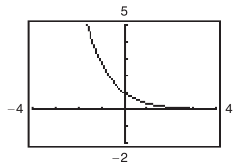
The graph has a hole at $x = 2$. The graph appears to be continuous, but the function is not continuous on $[-4, 4]$. It is not obvious from the graph that the function has a discontinuity at $x = 2$.

87. $f(x) = \dfrac{\ln(x^2 + 1)}{x}$



The graph has a hole at $x = 0$. The graph appears to be continuous, but the function is not continuous on $[-4, 4]$. It is not obvious from the graph that the function has a discontinuity at $x = 0$.

88. $f(x) = \dfrac{e^{-x} + 1}{e^x - 1}$



The graph has a hole at $x = 0$. The graph appears to be continuous, but the function is not continuous on $[-4, 4]$. It is not obvious from the graph that the function has a discontinuity at $x = 0$.

89. $f(x) = \frac{1}{12}x^4 - x^3 + 4$ is continuous on the interval $[1, 2]$. $f(1) = \frac{37}{12}$ and $f(2) = -\frac{8}{3}$. By the Intermediate Value Theorem, there exists a number c in $[1, 2]$ such that $f(c) = 0$.

90. $f(x) = -\dfrac{5}{x} + \tan\dfrac{\pi x}{10}$ is continuous on the interval $[1, 4]$.

$f(1) = -5 + \tan\dfrac{\pi}{10} \approx -4.7$ and

$f(4) = -\dfrac{5}{4} + \tan\dfrac{2\pi}{5} \approx 1.8$. By the Intermediate Value Theorem, there exists a number c in $[1, 4]$ such that $f(c) = 0$.

91. h is continuous on the interval $\left[0, \frac{\pi}{2}\right]$.

$h(0) = -2 < 0$ and $h\left(\frac{\pi}{2}\right) \approx 0.91 > 0$. By the Intermediate Value Theorem, there exists a number c in $\left[0, \frac{\pi}{2}\right]$ such that $h(c) = 0$.

92. g is continuous on the interval $[0, 1]$.

$g(0) \approx -2.77 < 0$ and $g(1) \approx 1.61 > 0$. By the Intermediate Value Theorem, there exists a number c in $[0, 1]$ such that $g(c) = 0$.

93. $f(x) = x^3 + x - 1$

$f(x)$ is continuous on $[0, 1]$.

$f(0) = -1$ and $f(1) = 1$

By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $f(x)$, you find that $x \approx 0.68$. Using the *root* feature, you find that $x \approx 0.6823$.

94. $f(x) = x^4 - x^2 + 3x - 1$

$f(x)$ is continuous on $[0, 1]$.

$f(0) = -1$ and $f(1) = 2$

By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $f(x)$, you find that $x \approx 0.37$. Using the *root* feature, you find that $x \approx 0.3733$.

95. $g(t) = 2 \cos t - 3t$

g is continuous on $[0, 1]$.

$g(0) = 2 > 0$ and $g(1) \approx -1.9 < 0$.

By the Intermediate Value Theorem, $g(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $g(t)$, you find that $t \approx 0.56$. Using the *root* feature, you find that $t \approx 0.5636$.

96. $h(\theta) = \tan \theta + 3\theta - 4$ is continuous on $[0, 1]$.

$h(0) = -4$ and $h(1) = \tan(1) - 1 \approx 0.557$.

By the Intermediate Value Theorem, $h(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $h(\theta)$, you find that $\theta \approx 0.91$. Using the *root* feature, you obtain $\theta \approx 0.9071$.

97. $f(x) = x + e^x - 3$

f is continuous on $[0, 1]$.

$f(0) = e^0 - 3 = -2 < 0$ and

$f(1) = 1 + e - 3 = e - 2 > 0$.

By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $f(x)$, you find that $x \approx 0.79$. Using the *root* feature, you find that $x \approx 0.7921$.

98. $g(x) = 5 \ln(x + 1) - 2$

g is continuous on $[0, 1]$.

$g(0) = 5 \ln(0 + 1) - 2 = -2$ and

$g(1) = 5 \ln(2) - 2 > 0$.

By the Intermediate Value Theorem, $g(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $g(x)$, you find that $x \approx 0.49$. Using the *root* feature, you find that $x \approx 0.4918$.

99. $f(x) = x^2 + x - 1$

f is continuous on $[0, 5]$.

$f(0) = -1$ and $f(5) = 29$

$-1 < 11 < 29$

The Intermediate Value Theorem applies.

$$x^2 + x - 1 = 11$$
$$x^2 + x - 12 = 0$$
$$(x + 4)(x - 3) = 0$$
$$x = -4 \text{ or } x = 3$$

$c = 3$ ($x = -4$ is not in the interval.)

So, $f(3) = 11$.

100. $f(x) = x^2 - 6x + 8$

f is continuous on $[0, 3]$.

$f(0) = 8$ and $f(3) = -1$

$-1 < 0 < 8$

The Intermediate Value Theorem applies.

$$x^2 - 6x + 8 = 0$$
$$(x - 2)(x - 4) = 0$$
$$x = 2 \text{ or } x = 4$$

$c = 2$ ($x = 4$ is not in the interval.)

So, $f(2) = 0$.

101. $f(x) = x^3 - x^2 + x - 2$

f is continuous on $[0, 3]$.

$f(0) = -2$ and $f(3) = 19$

$-2 < 4 < 19$

The Intermediate Value Theorem applies.

$$x^3 - x^2 + x - 2 = 4$$
$$x^3 - x^2 + x - 6 = 0$$
$$(x - 2)(x^2 + x + 3) = 0$$
$$x = 2$$

$c = 2$ $\left(x^2 + x + 3 \text{ has no real solution.}\right)$

So, $f(2) = 4$.

102. $f(x) = \dfrac{x^2 + x}{x - 1}$

f is continuous on $\left[\frac{5}{2}, 4\right]$. The nonremovable discontinuity, $x = 1$, lies outside the interval.

$f\left(\frac{5}{2}\right) = \frac{35}{6}$ and $f(4) = \frac{20}{3}$

$\frac{35}{6} < 6 < \frac{20}{3}$

The Intermediate Value Theorem applies.

$$\frac{x^2 + x}{x - 1} = 6$$
$$x^2 + x = 6x - 6$$
$$x^2 - 5x + 6 = 0$$
$$(x - 2)(x - 3) = 0$$
$$x = 2 \text{ or } x = 3$$

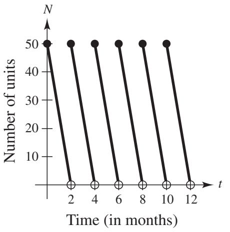
$c = 3$ $(x = 2$ is not in the interval.$)$

So, $f(3) = 6$.

103. $N(t) = 25\left(2\left[\!\left[\dfrac{t + 2}{2}\right]\!\right] - t\right)$

t	0	1	1.8	2	3	3.8
$N(t)$	50	25	5	50	25	5

There is a nonremovable discontinuity at every positive even integer. The company replenishes its inventory every two months.



104. $\lim_{t \to 4^-} f(t) \approx 28$

$\lim_{t \to 4^+} f(t) \approx 56$

At the end of day 3, the amount of chlorine in the pool has decreased to about 28 ounces. At the beginning of day 4, more chlorine was added, and the amount is now about 56 ounces.

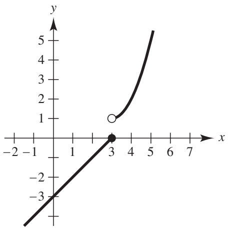
105. (a) The limit does not exist at $x = c$.

(b) The function is not defined at $x = c$.

(c) The limit exists at $x = c$, but it is not equal to the value of the function at $x = c$.

(d) The limit does not exist at $x = c$.

106. Answers will vary. Sample answer:



The function is not continuous at $x = 3$ because $\lim_{x \to 3^+} f(x) = 1 \neq 0 = \lim_{x \to 3^-} f(x)$.

107. If f and g are continuous for all real x, then so is $f + g$ (Theorem 1.11, part 2). However, f/g might not be continuous if $g(x) = 0$. For example, let $f(x) = x$ and $g(x) = x^2 - 1$. Then f and g are continuous for all real x, but f/g is not continuous at $x = \pm 1$.

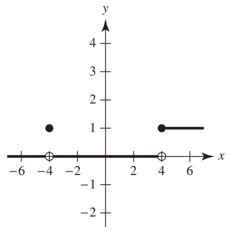
108. A discontinuity at c is removable if the function f can be made continuous at c by appropriately defining (or redefining) $f(c)$. Otherwise, the discontinuity is nonremovable.

(a) $f(x) = \dfrac{|x - 4|}{x - 4}$

(b) $f(x) = \dfrac{\sin(x + 4)}{x + 4}$

(c) $f(x) = \begin{cases} 1, & x \ge 4 \\ 0, & -4 < x < 4 \\ 1, & x = -4 \\ 0, & x < -4 \end{cases}$

$x = 4$ is nonremovable, $x = -4$ is removable



109. True

(1) $f(c) = L$ is defined.

(2) $\lim\limits_{x \to c} f(x) = L$ exists.

(3) $f(c) = \lim\limits_{x \to c} f(x)$

All of the conditions for continuity are met.

110. True. If $f(x) = g(x), x \ne c$, then $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} g(x)$ (if they exist) and at least one of these limits then does not equal the corresponding function value at $x = c$.

111. True. For $x \in (-1, 0), [\![x]\!] = -1$, which implies that $\lim\limits_{x \to 0^-} [\![x]\!] = -1$.

112. False. $f(1)$ is not defined and $\lim\limits_{x \to 1} f(x)$ does not exist.

113. Let $s(t)$ be the position function for the run up to the campsite. $s(0) = 0$ ($t = 0$ corresponds to 8:00 A.M., $s(20) = k$ (distance to campsite)). Let $r(t)$ be the position function for the run back down the mountain: $r(0) = k, r(10) = 0$. Let $f(t) = s(t) - r(t)$.

When $t = 0$ (8:00 A.M.), $f(0) = s(0) - r(0) = 0 - k < 0$.

When $t = 10$ (8:00 A.M.), $f(10) = s(10) - r(10) > 0$.

Because $f(0) < 0$ and $f(10) > 0$, there must be a value t in the interval $[0, 10]$ such that $f(t) = 0$. If $f(t) = 0$, then $s(t) - r(t) = 0$, which gives us $s(t) = r(t)$. Therefore, at some time t, where $0 \le t \le 10$, the position functions for the run up and the run down are equal.

114. Let $V = \frac{4}{3}\pi r^3$ be the volume of a sphere with radius r.

V is continuous on $[5, 8]$. $V(5) = \dfrac{500\pi}{3} \approx 523.6$ and $V(8) = \dfrac{2048\pi}{3} \approx 2144.7$. Because $523.6 < 1500 < 2144.7$, the Intermediate Value Theorem guarantees that there is at least one value r between 5 and 8 such that $V(r) = 1500$. (In fact, $r \approx 7.1012$.)

115. Let c be any real number. Then $\lim\limits_{x \to c} f(x)$ does not exist because there are both rational and irrational numbers arbitrarily close to c. Therefore, f is not continuous at c.

116. If $x = 0$, then $f(0) = 0$ and $\lim\limits_{x \to 0} f(x) = 0$. So, f is continuous at $x = 0$.

If $x \ne 0$, then $\lim\limits_{t \to x} f(t) = 0$ for x rational, whereas $\lim\limits_{t \to x} f(t) = \lim\limits_{t \to x} kt = kx \ne 0$ for x irrational. So, f is not continuous for all $x \ne 0$.

117. $f(x) = \begin{cases} 1 - x^2, & x \le c \\ x, & x > c \end{cases}$

f is continuous for $x < c$ and for $x > c$. At $x = c$, you need $1 - c^2 = c$. Solving $c^2 + c - 1$, you obtain

$$c = \frac{-1 \pm \sqrt{1 + 4}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$

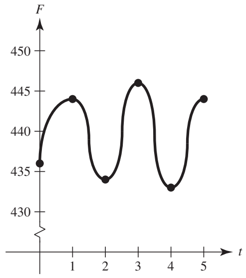
118. $f(x) = \dfrac{\sqrt{x + c^2} - c}{x}, c > 0$

Domain: $x + c^2 \geq 0 \Rightarrow x \geq -c^2$ and $x \neq 0, \left[-c^2, 0\right) \cup (0, \infty)$

$$\lim_{x\to 0} \frac{\sqrt{x + c^2} - c}{x} = \lim_{x\to 0} \frac{\sqrt{x + c^2} - c}{x} \cdot \frac{\sqrt{x + c^2} + c}{\sqrt{x + c^2} + c} = \lim_{x\to 0} \frac{\left(x + c^2\right) - c^2}{x\left[\sqrt{x + c^2} + c\right]} = \lim_{x\to 0} \frac{1}{\sqrt{x + c^2} + c} = \frac{1}{2c}$$

Define $f(0) = 1/(2c)$ to make f continuous at $x = 0$.

119. (a)



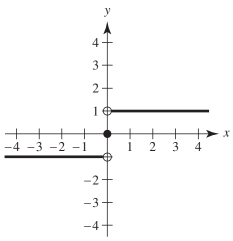
(b) No. The frequency is oscillating.

120. $\operatorname{sgn}(x) = \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$

(a) $\displaystyle\lim_{x\to 0^-} \operatorname{sgn}(x) = -1$

(b) $\displaystyle\lim_{x\to 0^+} \operatorname{sgn}(x) = 1$

(c) $\displaystyle\lim_{x\to 0} \operatorname{sgn}(x)$ does not exist.



121. The functions agree for integer values of x:

$$\left.\begin{aligned} g(x) &= 3 - [\![-x]\!] = 3 - (-x) = 3 + x \\ f(x) &= 3 + [\![x]\!] = 3 + x \end{aligned}\right\} \text{for } x \text{ an integer}$$

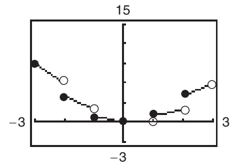
However, for non-integer values of x, the functions differ by 1.

$f(x) = 3 + [\![x]\!] = g(x) - 1 = 2 - [\![-x]\!]$.

For example,

$f\left(\frac{1}{2}\right) = 3 + 0 = 3, g\left(\frac{1}{2}\right) = 3 - (-1) = 4.$

122. $h(x) = x[\![x]\!]$



h has nonremovable discontinuities at $x = \pm1, \pm2, \pm3, \ldots$.

123. (1) $f(c)$ is defined.

(2) $\displaystyle\lim_{x\to c} f(x) = \lim_{\Delta x\to 0} f(c + \Delta x) = f(c)$ exists.

[Let $x = c + \Delta x$. As $x \to c, \Delta x \to 0$]

(3) $\displaystyle\lim_{x\to c} f(x) = f(c)$.

Therefore, f is continuous at $x = c$.

124. Suppose there exists x_1 in $[a, b]$ such that $f(x_1) > 0$ and there exists x_2 in $[a, b]$ such that $f(x_2) < 0$. Then by the Intermediate Value Theorem, $f(x)$ must equal zero for some value of x in $[x_1, x_2]$ (or $[x_2, x_1]$ if $x_2 < x_1$). So, f would have a zero in $[a, b]$, which is a contradiction. Therefore, $f(x) > 0$ for all x in $[a, b]$ or $f(x) < 0$ for all x in $[a, b]$.

125. Let y be a real number. If $y = 0$, then $x = 0$. If $y > 0$, then let $0 < x_0 < \pi/2$ such that $M = \tan x_0 > y$ (this is possible because the tangent function increases without bound on $[0, \pi/2)$). By the Intermediate Value Theorem, $f(x) = \tan x$ is continuous on $[0, x_0]$ and $0 < y < M$, which implies that there exists x between 0 and x_0 such that $\tan x = y$. The argument is similar when $y < 0$.

126. (a) Define $f(x) = f_2(x) - f_1(x)$. Because f_1 and f_2 are continuous on $[a, b]$, so is f.

$f(a) = f_2(a) - f_1(a) > 0$ and $f(b) = f_2(b) - f_1(b) < 0$

By the Intermediate Value Theorem, there exists c in $[a, b]$ such that $f(c) = 0$.

$f(c) = f_2(c) - f_1(c) = 0 \Rightarrow f_1(c) = f_2(c)$

(b) Let $f_1(x) = x$ and $f_2(x) = \cos x$, continuous on $[0, \pi/2]$, $f_1(0) < f_2(0)$ and $f_1(\pi/2) > f_2(\pi/2)$.

So by part (a), there exists c in $[0, \pi/2]$ such that $c = \cos c$.

Using a graphing utility, $c \approx 0.739$.

127. The domain of $f(x) = \dfrac{3}{\sqrt{x^2 - 1}}$ is all x where $x^2 - 1 > 0 \Rightarrow x^2 > 1 \Rightarrow x > 1$ or $x < -1$.

So, f is continuous on $(1, \infty) \cup (-\infty, -1)$. So, the answer is A.

128. $f(x) = \begin{cases} \dfrac{x^3 + x - 2}{x - 1}, & x \neq 1 \\ c, & x = 1 \end{cases}$ has a possible discontinuity at $x = 1$.

$f(1) = c$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^3 + x - 2}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x^2 + x + 2)}{x - 1} = \lim_{x \to 1} (x^2 + x + 2) = 4$$

$f(x)$ is continuous at $x = 1$ when $f(1) = \lim_{x \to 1} f(x) \Rightarrow c = 4$.

So, the answer is D.

129. (a) $p(x) = \begin{cases} 2, & x \le -1 \\ ax + b, & -1 < x < 3 \\ -2, & x \ge 3 \end{cases}$

$\lim_{x \to -1^+} p(x) = \lim_{x \to -1^+} (ax + b) = a(-1) + b = -a + b$

$\lim_{x \to 3^-} p(x) = \lim_{x \to 3^-} (ax + b) = a(3) + b = 3a + b$

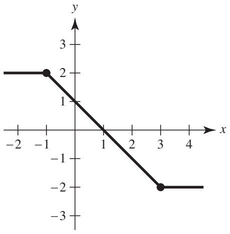
$p(x)$ is continuous when $-a + b = 2$ and

$3a + b = -2$

$$\begin{array}{rcrcr} -a & + & b & = & 2 \\ 3a & + & b & = & -2 \\ \hline -4a & & & = & 4 \\ a & & & = & -1 \end{array} \qquad \begin{array}{r} -a + b = 2 \\ -(-1) + b = 2 \\ b = 1 \end{array}$$

So, p is continuous when $a = -1$ and $b = 1$.

(b)



(c) When $a = -1$ and $b = 1$, $ax + b = -x + 1$.

So, $\lim_{x \to 0} p(x) = \lim_{x \to 0} (-x + 1) = -(0) + 1 = 1$.

130. $f(x) = \begin{cases} x^2 + 5, & x \le 2 \\ \dfrac{x^4 - 16}{x^2 - 4}, & x > 2 \end{cases}$

(a) $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \dfrac{x^4 - 16}{x^2 - 4} = \lim_{x \to 2^+} \dfrac{(x^2 - 4)(x^2 + 4)}{x^2 - 4} = \lim_{x \to 2^+} (x^2 + 4) = (2)^2 + 4 = 8$

(b) $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^2 + 5) = 2^2 + 5 = 9$

(c) $\lim_{x \to 2} f(x)$ does not exist because $\lim_{x \to 2^+} f(x) \neq \lim_{x \to 2^-} f(x)$.

(d) $\lim_{x \to -2} f(x) = \lim_{x \to -2} (x^2 + 5) = (-2)^2 + 5 = 9$

Section 1.5 Infinite Limits

1. $\lim_{x\to-2^+} 2\left|\dfrac{x}{x^2-4}\right| = \infty$

$\lim_{x\to-2^-} 2\left|\dfrac{x}{x^2-4}\right| = \infty$

2. $\lim_{x\to-2^+} \dfrac{1}{x+2} = \infty$

$\lim_{x\to-2^-} \dfrac{1}{x+2} = -\infty$

3. $\lim_{x\to-2^+} \tan\dfrac{\pi x}{4} = -\infty$

$\lim_{x\to-2^-} \tan\dfrac{\pi x}{4} = \infty$

4. $\lim_{x\to-2^+} \sec\dfrac{\pi x}{4} = \infty$

$\lim_{x\to-2^-} \sec\dfrac{\pi x}{4} = -\infty$

5. $f(x) = \dfrac{1}{x-4}$

As x approaches 4 from the left, $x - 4$ is a small negative number. So,

$\lim_{x\to4^-} f(x) = -\infty.$

As x approaches 4 from the right, $x - 4$ is a small positive number. So,

$\lim_{x\to4^+} f(x) = \infty.$

6. $f(x) = \dfrac{-1}{x-4}$

As x approaches 4 from the left, $x - 4$ is a small negative number. So,

$\lim_{x\to4^-} f(x) = \infty.$

As x approaches 4 from the right, $x - 4$ is a small positive number. So,

$\lim_{x\to4^+} f(x) = -\infty.$

7. $f(x) = \dfrac{1}{(x-4)^2}$

As x approaches 4 from the left or right, $(x-4)^2$ is a small positive number. So,

$\lim_{x\to4^+} f(x) = \lim_{x\to4^-} f(x) = \infty.$

8. $f(x) = \dfrac{-1}{(x-4)^2}$

As x approaches 4 from the left or right, $(x-4)^2$ is a small positive number. So,

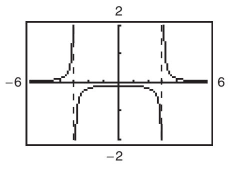
$\lim_{x\to4^-} f(x) = \lim_{x\to4^+} f(x) = -\infty.$

9. $f(x) = \dfrac{1}{x^2-9}$

x	–3.5	–3.1	–3.01	–3.001	–3	–2.999	–2.99	–2.9	–2.5
$f(x)$	0.308	1.639	16.64	166.6	?	–166.7	–16.69	–1.695	–0.364

$\lim_{x\to-3^-} f(x) = \infty$

$\lim_{x\to-3^+} f(x) = -\infty$

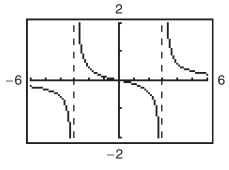


10. $f(x) = \dfrac{x}{x^2-9}$

x	–3.5	–3.1	–3.01	–3.001	–3	–2.999	–2.99	–2.9	–2.5
$f(x)$	–1.077	–5.082	–50.08	–500.1	?	499.9	49.92	4.915	0.9091

$\lim_{x\to-3^-} f(x) = -\infty$

$\lim_{x\to-3^+} f(x) = \infty$

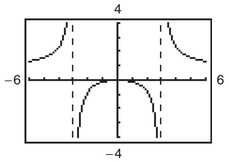


11. $f(x) = \dfrac{x^2}{x^2 - 9}$

x	-3.5	-3.1	-3.01	-3.001	-3	-2.999	-2.99	-2.9	-2.5
$f(x)$	3.769	15.75	150.8	1501	?	-1499	-149.3	-14.25	-2.273

$\lim\limits_{x\to -3^-} f(x) = \infty$

$\lim\limits_{x\to -3^+} f(x) = -\infty$

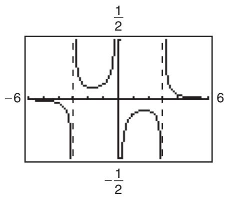


12. $f(x) = \dfrac{1}{x^3 - 9x}$

x	-3.5	-3.1	-3.01	-3.001	-3	-2.999	-2.99	-2.9	-2.5
$f(x)$	-0.088	-0.529	-5.528	-55.528	?	55.583	5.583	0.584	0.145

$\lim\limits_{x\to -3^-} f(x) = -\infty$

$\lim\limits_{x\to -3^+} f(x) = \infty$

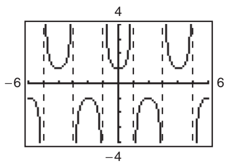


13. $f(x) = \sec\dfrac{\pi x}{2}$

x	-3.5	-3.1	-3.01	-3.001	-3	-2.999	-2.99	-2.9	-2.5
$f(x)$	1.414	6.392	63.665	636.620	?	-636.620	-63.665	-6.392	-1.414

$\lim\limits_{x\to -3^-} f(x) = \infty$

$\lim\limits_{x\to -3^+} f(x) = -\infty$

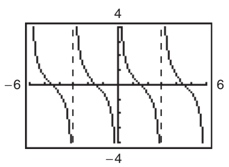


14. $f(x) = \cot\dfrac{\pi x}{3}$

x	-3.5	-3.1	-3.01	-3.001	-3	-2.999	-2.99	-2.9	-2.5
$f(x)$	-1.7321	-9.514	-95.49	-954.9	?	954.9	95.49	9.514	1.7321

$\lim\limits_{x\to -3^-} f(x) = -\infty$

$\lim\limits_{x\to -3^+} f(x) = \infty$



15. $f(x) = \dfrac{1}{x^2}$

$\lim\limits_{x\to 0^+} \dfrac{1}{x^2} = \infty = \lim\limits_{x\to 0^-} \dfrac{1}{x^2}$

Therefore, $x = 0$ is a vertical asymptote.

16. $f(x) = \dfrac{2}{(x-3)^3}$

$\lim\limits_{x\to 3^-} \dfrac{2}{(x-3)^3} = -\infty$

$\lim\limits_{x\to 3^+} \dfrac{2}{(x-3)^3} = \infty$

Therefore, $x = 3$ is a vertical asymptote.

17. $f(x) = \dfrac{x^2}{x^2-4} = \dfrac{x^2}{(x+2)(x-2)}$

$\lim\limits_{x\to -2^-} \dfrac{x^2}{x^2-4} = \infty$ and $\lim\limits_{x\to -2^+} \dfrac{x^2}{x^2-4} = -\infty$

Therefore, $x = -2$ is a vertical asymptote.

$\lim\limits_{x\to 2^-} \dfrac{x^2}{x^2-4} = -\infty$ and $\lim\limits_{x\to 2^+} \dfrac{x^2}{x^2-4} = \infty$

Therefore, $x = 2$ is a vertical asymptote.

18. $f(x) = \dfrac{3x}{x^2+9}$

No vertical asymptotes because the denominator is never zero.

19. $f(x) = \dfrac{x-3}{x^2+3x} = \dfrac{x-3}{x(x+3)}$

$\lim\limits_{x\to -3^-} \dfrac{x-3}{x^2+3x} = -\infty$ and $\lim\limits_{x\to -3^+} \dfrac{x-3}{x^2+3x} = \infty.$

Therefore, $x = -3$ is a vertical asymptote.

$\lim\limits_{x\to 0^-} \dfrac{x-3}{x^2+3x} = \infty$ and $\lim\limits_{x\to -0^+} \dfrac{x-3}{x^2+3x} = \infty.$

Therefore, $x = 0$ is a vertical asymptote.

20. $h(s) = \dfrac{3s+4}{s^2-16} = \dfrac{3s+4}{(s-4)(s+4)}$

$\lim\limits_{s\to 4^-} \dfrac{3s+4}{s^2-16} = -\infty$ and $\lim\limits_{s\to 4^+} \dfrac{3s+4}{s^2-16} = \infty$

Therefore, $s = 4$ is a vertical asymptote.

$\lim\limits_{s\to -4^-} \dfrac{3s+4}{s^2-16} = -\infty$ and $\lim\limits_{s\to -4^+} \dfrac{3s+4}{s^2-16} = \infty$

Therefore, $s = -4$ is a vertical asymptote.

21. $f(x) = \dfrac{3}{x^2+x-2} = \dfrac{3}{(x+2)(x-1)}$

$\lim\limits_{x\to -2^-} \dfrac{3}{x^2+x-2} = \infty$ and $\lim\limits_{x\to -2^+} \dfrac{3}{x^2+x-2} = -\infty$

Therefore, $x = -2$ is a vertical asymptote.

$\lim\limits_{x\to 1^-} \dfrac{3}{x^2+x-2} = -\infty$ and $\lim\limits_{x\to 1^+} \dfrac{3}{x^2+x-2} = \infty$

Therefore, $x = 1$ is a vertical asymptote.

22. $g(x) = \dfrac{x^3-8}{x-2} = \dfrac{(x-2)(x^2+2x+4)}{x-2}$

$= x^2 + 2x + 4,\ x \neq 2$

$\lim\limits_{x\to 2} g(x) = 4 + 4 + 4 = 12$

There are no vertical asymptotes. The graph has a hole at $x = 2$.

23. $f(x) = \dfrac{x^2-2x-15}{x^3-5x^2+x-5}$

$= \dfrac{(x-5)(x+3)}{(x-5)(x^2+1)}$

$= \dfrac{x+3}{x^2+1},\ x \neq 5$

$\lim\limits_{x\to 5} f(x) = \dfrac{5+3}{5^2+1} = \dfrac{15}{26}$

There are no vertical asymptotes. The graph has a hole at $x = 5$.

24. $h(x) = \dfrac{x^2-9}{x^3+3x^2-x-3}$

$= \dfrac{(x-3)(x+3)}{(x-1)(x+1)(x+3)}$

$= \dfrac{x-3}{(x+1)(x-1)},\ x \neq -3$

$\lim\limits_{x\to -1^-} h(x) = -\infty$ and $\lim\limits_{x\to -1^+} h(x) = \infty$

Therefore, $x = -1$ is a vertical asymptote.

$\lim\limits_{x\to 1^-} h(x) = \infty$ and $\lim\limits_{x\to 1^+} h(x) = -\infty$

Therefore, $x = 1$ is a vertical asymptote.

$\lim\limits_{x\to -3} h(x) = \dfrac{-3-3}{(-3+1)(-3-1)} = -\dfrac{3}{4}$

Therefore, the graph has a hole at $x = -3$.

25. $f(x) = \dfrac{e^{-2x}}{x-1}$

$\lim\limits_{x\to 1^-} f(x) = -\infty$ and $\lim\limits_{x\to 1^+} = \infty$

Therefore, $x = 1$ is a vertical asymptote.

26. $g(x) = xe^{-2x}$

The function is continuous for all x. Therefore, there are no vertical asymptotes.

27. $h(t) = \dfrac{\ln(t^2+1)}{t+2}$

$\lim\limits_{t\to -2^-} h(t) = -\infty$ and $\lim\limits_{t\to -2^+} = \infty$

Therefore, $t = -2$ is a vertical asymptote.

28. $f(z) = \ln(z^2 - 4) = \ln[(z+2)(z-2)]$
$= \ln(z+2) + \ln(z-2)$

The function is undefined for $-2 < z < 2$.

Therefore, the graph has holes at $z = \pm 2$, and no vertical asymptotes.

29. $f(x) = \dfrac{1}{e^x - 1}$

$\lim_{x\to 0^-} f(x) = -\infty$ and $\lim_{x\to 0^+} f(x) = \infty$

Therefore, $x = 0$ is a vertical asymptote.

30. $f(x) = \ln(x+3)$

$\lim_{x\to -3} f(x) = -\infty$

Therefore, $x = -3$ is a vertical asymptote.

31. $f(x) = \csc \pi x = \dfrac{1}{\sin \pi x}$

Let n be any integer.

$\lim_{x\to n} f(x) = -\infty$ or ∞

Therefore, the graph has vertical asymptotes at $x = n$, where n is an integer.

32. $f(x) = \tan \pi x = \dfrac{\sin \pi x}{\cos \pi x}$

$\cos \pi x = 0$ for $x = \dfrac{2n+1}{2}$, where n is an integer.

$\lim_{x\to \frac{2n+1}{2}} f(x) = \infty$ or $-\infty$

Therefore, the graph has vertical asymptotes at $x = \dfrac{2n+1}{2}$, where n is an integer.

33. $s(t) = \dfrac{t}{\sin t}$

$\sin t = 0$ for $t = n\pi$, where n is an integer.

$\lim_{t\to n\pi} s(t) = \infty$ or $-\infty$ (for $n \neq 0$)

Therefore, the graph has vertical asymptotes at $t = n\pi$, for $n \neq 0$.

$\lim_{t\to 0} s(t) = 1$

Therefore, the graph has a hole at $t = 0$.

34. $g(\theta) = \dfrac{\tan \theta}{\theta} = \dfrac{\sin \theta}{\theta \cos \theta}$

$\cos \theta = 0$ for $\theta = \dfrac{\pi}{2} + n\pi$, where n is an integer.

$\lim_{\theta\to \frac{\pi}{2}+n\pi} g(\theta) = \infty$ or $-\infty$

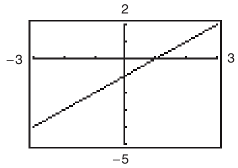
Therefore, the graph has vertical asymptotes at $\theta = \dfrac{\pi}{2} + n\pi$.

$\lim_{\theta\to 0} g(\theta) = 1$

Therefore, the graph has a hole at $\theta = 0$.

35. $\lim_{x\to -1} \dfrac{x^2 - 1}{x+1} = \lim_{x\to -1} (x-1) = -2$

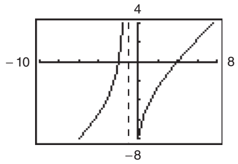
Removable discontinuity at $x = -1$



36. $\lim_{x\to -1^-} \dfrac{x^2 - 2x - 8}{x+1} = \infty$

$\lim_{x\to -1^+} \dfrac{x^2 - 2x - 8}{x+1} = -\infty$

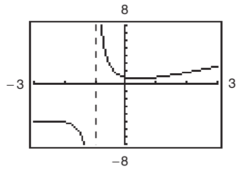
Vertical asymptote at $x = -1$



37. $\lim_{x\to -1^+} \dfrac{x^2 + 1}{x+1} = \infty$

$\lim_{x\to -1^-} \dfrac{x^2 + 1}{x+1} = -\infty$

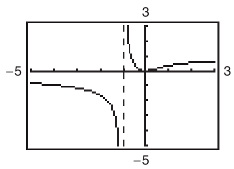
Vertical asymptote at $x = -1$



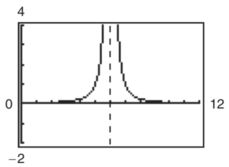
38. $\lim_{x\to -1^+} \dfrac{\ln(x^2 + 1)}{x+1} = \infty$

$\lim_{x\to -1^-} \dfrac{\ln(x^2 + 1)}{x+1} = -\infty$

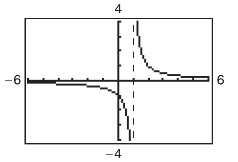
Vertical asymptote at $x = -1$



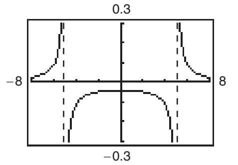
39.



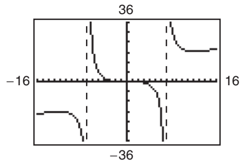
$\lim_{x\to 6^+} \dfrac{1}{(x-6)^2} = \infty$

40. 

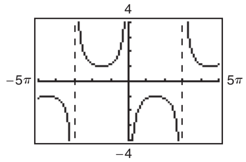
$$\lim_{x\to 1^-} \frac{x^2 + x + 1}{x^3 - 1} = -\infty$$

41. 

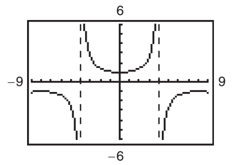
$$\lim_{x\to 5^-} \frac{1}{x^2 - 25} = -\infty$$

42. 

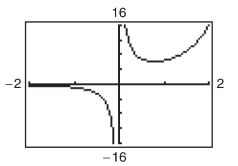
$$\lim_{x\to 7^+} \frac{x^3}{x^2 - 49} = \infty$$

43. 

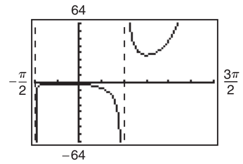
$$\lim_{x\to (3\pi)^+} \left(-\csc \frac{x}{3}\right) = \infty$$

44. 

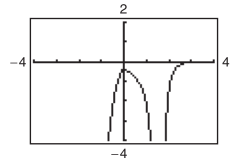
$$\lim_{x\to 4^+} \sec\frac{\pi x}{8} = -\infty$$

45. 

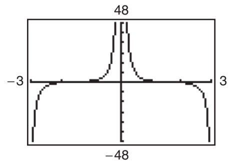
$$\lim_{x\to 0^-} \frac{2e^x}{\sin x} = -\infty$$

46. 

$$\lim_{x\to (\pi/2)^+} \frac{-2e^x}{\cos x} = \infty$$

47. 

$$\lim_{x\to 0^+} \frac{e^{-x}}{\ln|\sin x|} = 0$$

48. 

$$\lim_{x\to 0^-} \frac{-\cos x}{\ln|\cos x|} = \infty$$

49. $\displaystyle\lim_{x\to -1^+} \frac{1}{x + 1} = \infty$

50. $\displaystyle\lim_{x\to 1^-} \frac{-1}{(x - 1)^2} = -\infty$

51. $\displaystyle\lim_{x\to 2^+} \frac{x}{x - 2} = \infty$

52. $\displaystyle\lim_{x\to 2^-} \frac{x^2}{2x - x^2} = \lim_{x\to 2^-} \frac{x^2}{x(2 - x)} = \lim_{x\to 2^-} \frac{x}{2 - x} = \infty$

53. $$\begin{aligned}\lim_{x\to 2^-} \frac{x + 3}{x^2 + x - 6} &= \lim_{x\to 2^-} \frac{x + 3}{(x + 3)(x - 2)}\\ &= \lim_{x\to 2^-} \frac{1}{x - 2}\\ &= -\infty\end{aligned}$$

54. $$\begin{aligned}\lim_{x\to (3/2)^+} \frac{6x^2 + x - 1}{4x^2 - 4x - 3} &= \lim_{x\to (3/2)^+} \frac{(3x - 1)(2x + 1)}{(2x - 3)(2x + 1)}\\ &= \lim_{x\to (3/2)^+} \frac{3x - 1}{2x - 3}\\ &= \infty\end{aligned}$$

55. $\displaystyle\lim_{x\to 0^-} \left(1 + \frac{1}{x}\right) = -\infty$

56. $\lim_{x\to 0^+}\left(6 - \frac{1}{x^3}\right) = -\infty$

57. $\lim_{x\to -4^-}\left(x^2 + \frac{2}{x+4}\right) = -\infty$

58. $\lim_{x\to 2^+}\left(\frac{x}{3} + \cot\frac{\pi x}{2}\right) = \frac{2}{3} + \infty = \infty$

59. $\lim_{x\to 0^+}\frac{3x+2}{\tan \pi x} = \infty$

60. $\lim_{x\to(\pi/2)^+}\frac{x^2-2}{\sec x} = \lim_{x\to(\pi/2)^+}\left[(x^2-2)(\cos x)\right]$

$$= \left[\left(\frac{\pi}{2}\right)^2 - 2\right]\left(\cos\frac{\pi}{2}\right)$$

$$= 0$$

61. $\lim_{x\to 8^-}\frac{e^x}{(x-8)^3} = -\infty$

62. $\lim_{x\to \pi^+}\frac{e^{-0.5x}}{\sin(-x)} = \infty$

63. $\lim_{x\to 0^+} 16\ln x = (16)(-\infty) = -\infty$

64. $\lim_{x\to 4^+}\ln(x^2 - 16) = -\infty$

65. $\lim_{x\to(\pi/2)^-} x\sec x = \left(\frac{\pi}{2}\right)(\infty) = \infty$

66. $\lim_{x\to(1/2)^+} x^2\tan \pi x = -\infty$

67. The numerator is factorable.

$$h(x) = \frac{3x+15}{x^2-25} = \frac{3(x+5)}{(x+5)(x-5)} = \frac{3}{x-5}$$

So, the graph of h has a vertical asymptote at $x = 5$, and a removable discontinuity at $x = -5$.

68. The limits should have been subtracted.

Because $\lim_{x\to -1^-}\frac{1}{1+x^2} = \frac{1}{2}$ and $\lim_{x\to -1^-}\tan\frac{\pi x}{2} = \infty$,

$$\lim_{x\to -1^-}\left(\frac{1}{1+x^2} - \tan\frac{\pi x}{2}\right) = \frac{1}{2} - \infty = -\infty.$$

69. A limit in which $f(x)$ increases or decreases without bound as x approaches c is called an infinite limit. ∞ is not a number. Rather, the symbol

$\lim_{x\to c} f(x) = \infty$

says how the limit fails to exist.

70. No. For example, $f(x) = \frac{1}{x^2+1}$ has no vertical asymptote.

71. Because $f(x)$ has a vertical asymptote at $x = a$, each of these one-sided limits is either ∞ or $-\infty$. To determine which, evaluate $f(x)$ at a value close to $x = a$ from the right and a value close to $x = a$ from the left. A positive result means this one-sided limit approaches ∞, and a negative result means this one-sided limit approaches $-\infty$. These one-sided limits could also be examined graphically or numerically by using a table.

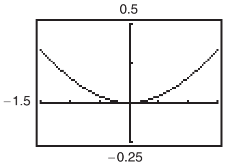
72. $\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-}\frac{1}{f(x)}$

When x approaches 2 from the left, the graph shows that the denominator $f(x)$ is positive and approaches 0. So, the quotient $\frac{1}{f(x)}$ yields an arbitrarily large positive number, which means that $\lim_{x\to 2^-} g(x) = \infty$.

When x approaches 2 from the right, the graph shows that the denominator $f(x)$ is negative and approaches 0. So, the quotient $\frac{1}{f(x)}$ yields an arbitrarily small negative number, which means that $\lim_{x\to 2^-} g(x) = -\infty$.

73. (a)

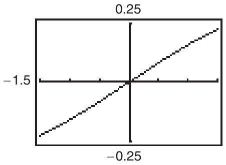
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.0411	0.0067	0.0017	≈ 0	≈ 0	≈ 0



$$\lim_{x\to 0^+} \frac{x - \sin x}{x} = 0$$

(b)

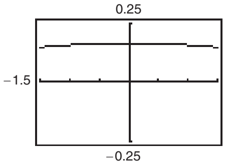
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.0823	0.0333	0.0167	0.0017	≈ 0	≈ 0



$$\lim_{x\to 0^+} \frac{x - \sin x}{x^2} = 0$$

(c)

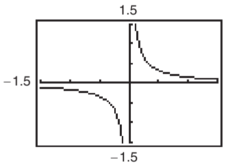
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.1646	0.1663	0.1666	0.1667	0.1667	0.1667



$$\lim_{x\to 0^+} \frac{x - \sin x}{x^3} = 0.1667\ (1/6)$$

(d)

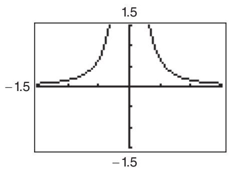
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.3292	0.8317	1.6658	16.67	166.7	1667.0



$$\lim_{x\to 0^+} \frac{x - \sin x}{x^4} = \infty \text{ or } n > 3, \lim_{x\to 0^+} \frac{x - \sin x}{x^n} = \infty.$$

(e)

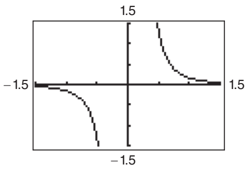
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.6584	4.1583	16.6583	1666.7	1.67×10^5	1.67×10^7



$$\lim_{x\to 0^+} \frac{x - \sin x}{x^5} = \infty$$

(f)

x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	1.3168	20.7917	166.58	1.67×10^5	1.67×10^8	1.67×10^{11}



$$\lim_{x\to 0^+} \frac{x - \sin x}{x^6} = \infty$$

When the power of x in the denominator is greater than 3, the limit is ∞.

74. $\lim_{V\to 0^+} P = \infty$

As the volume of the gas decreases, the pressure increases.

75. (a) $r = \dfrac{2(7)}{\sqrt{625 - 49}} = \dfrac{7}{12}$ ft/sec

(b) $r = \dfrac{2(15)}{\sqrt{625 - 225}} = \dfrac{3}{2}$ ft/sec

(c) $\lim_{x\to 25^-} \dfrac{2x}{\sqrt{625 - x^2}} = \infty$

As the distance of the base from the house approaches the length of the ladder, the rate increases without bound.

76. (a) Average speed $= \dfrac{\text{Total distance}}{\text{Total time}}$

$$50 = \frac{2d}{(d/x) + (d/y)}$$

$$50 = \frac{2xy}{y + x}$$

$$50y + 50x = 2xy$$

$$50x = 2xy - 50y$$

$$50x = 2y(x - 25)$$

$$\frac{25x}{x - 25} = y$$

Domain: $(25, \infty)$

(b)

x	30	40	50	60
y	150	66.667	50	42.857

(c) $\lim_{x\to 25^+} \dfrac{25x}{\sqrt{x - 25}} = \infty$

As x gets close to 25 mi/h, y becomes larger and larger.

77. True. The function is undefined at a vertical asymptote.

78. True

79. False. The graphs of $y = \tan x$, $y = \cot x$, $y = \sec x$, and $y = \csc x$ have vertical asymptotes.

80. False. Let

$$f(x) = \begin{cases} \dfrac{1}{x}, & x \neq 0 \\ 3, & x = 0. \end{cases}$$

The graph of f has a vertical asymptote at $x = 0$, but $f(0) = 3$.

81. Let $f(x) = \dfrac{1}{x^2}$ and $g(x) = \dfrac{1}{x^4}$, and $c = 0$.

$\lim_{x\to 0} \dfrac{1}{x^2} = \infty$ and $\lim_{x\to 0} \dfrac{1}{x^4} = \infty$, but

$$\lim_{x\to 0}\left(\frac{1}{x^2} - \frac{1}{x^4}\right) = \lim_{x\to 0}\left(\frac{x^2 - 1}{x^4}\right) = -\infty \neq 0.$$

82. Given $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = L$:

(1) Difference:

Let $h(x) = -g(x)$. Then $\lim_{x\to c} h(x) = -L$, and $\lim_{x\to c}\left[f(x) - g(x)\right] = \lim_{x\to c}\left[f(x) + h(x)\right] = \infty$, by the Sum Property.

(2) Product:

If $L > 0$, then for $\varepsilon = L/2 > 0$ there exists $\delta_1 > 0$ such that $\left|g(x) - L\right| < L/2$ whenever $0 < \left|x - c\right| < \delta_1$.

So, $L/2 < g(x) < 3L/2$. Because $\lim_{x\to c} f(x) = \infty$ then for $M > 0$, there exists $\delta_2 > 0$ such that $f(x) > M(2/L)$ whenever $\left|x - c\right| < \delta_2$. Let δ be the smaller of δ_1 and δ_2. Then for $0 < \left|x - c\right| < \delta$,

you have $f(x)g(x) > M(2/L)(L/2) = M$. Therefore $\lim_{x\to c} f(x)g(x) = \infty$. The proof is similar for $L < 0$.

(3) Quotient: Let $\varepsilon > 0$ be given.

There exists $\delta_1 > 0$ such that $f(x) > 3L/2\varepsilon$ whenever $0 < \left|x - c\right| < \delta_1$ and there exists $\delta_2 > 0$ such that $\left|g(x) - L\right| < L/2$ whenever $0 < \left|x - c\right| < \delta_2$. This inequality gives us $L/2 < g(x) < 3L/2$. Let δ be the smaller of δ_1 and δ_2. Then for $0 < \left|x - c\right| < \delta$, you have

$$\left|\frac{g(x)}{f(x)}\right| < \frac{3L/2}{3L/2\varepsilon} = \varepsilon.$$

Therefore, $\lim_{x\to c} \dfrac{g(x)}{f(x)} = 0$.

83. Given $\lim_{x\to c} f(x) = \infty$, let $g(x) = 1$. Then $\lim_{x\to c} \dfrac{g(x)}{f(x)} = 0$ by Theorem 1.15.

84. Given $\lim_{x\to c} \dfrac{1}{f(x)} = 0$. Suppose $\lim_{x\to c} f(x)$ exists and equals L.

Then, $\lim_{x\to c} \dfrac{1}{f(x)} = \dfrac{\lim_{x\to c} 1}{\lim_{x\to c} f(x)} = \dfrac{1}{L} = 0$.

This is not possible. So, $\lim_{x\to c} f(x)$ does not exist.

85. $f(x) = \dfrac{1}{x - 3}$ is defined for all $x > 3$.

Let $M > 0$ be given. You need $\delta > 0$ such that

$f(x) = \dfrac{1}{x - 3} > M$ whenever $3 < x < 3 + \delta$.

Equivalently, $x - 3 < \dfrac{1}{M}$ whenever

$\left|x - 3\right| < \delta, x > 3$.

So take $\delta = \dfrac{1}{M}$. Then for $x > 3$ and

$\left|x - 3\right| < \delta, \dfrac{1}{x - 3} > \dfrac{1}{8} = M$ and so $f(x) > M$.

86. $f(x) = \dfrac{1}{x - 5}$ is defined for all $x < 5$. Let $N < 0$ be given. You need $\delta > 0$ such that $f(x) = \dfrac{1}{x - 5} < N$ whenever $5 - \delta < x < 5$. Equivalently, $x - 5 > \dfrac{1}{N}$ whenever $\left|x - 5\right| < \delta, x < 5$. Equivalently, $\dfrac{1}{\left|x - 5\right|} < -\dfrac{1}{N}$ whenever $\left|x - 5\right| < \delta, x < 5$. So take $\delta = -\dfrac{1}{N}$. Note that $\delta > 0$ because $N < 0$. For $\left|x - 5\right| < \delta$ and $x < 5, \dfrac{1}{\left|x - 5\right|} > \dfrac{1}{\delta} = -N$, and $\dfrac{1}{x - 5} = -\dfrac{1}{\left|x - 5\right|} < N$.

87. $f(x) = \dfrac{5(x^2 - 4)}{2x^2 - 5x + 2} = \dfrac{5(x + 2)(x - 2)}{(2x - 1)(x - 2)} = \dfrac{5(x + 2)}{2x - 1}$

$f(x)$ has a removable discontinuity at $x = 2$, and a vertical asymptote at $x = \dfrac{1}{2}$. So, the answer is A.

88.
$$\begin{aligned}\lim_{x\to -1^-}\left(\sec\frac{\pi x}{2} + 2x\right) &= \lim_{x\to -1^-} \sec\frac{\pi x}{2} + \lim_{x\to -1^-} 2x\\ &= -\infty + (-2)\\ &= -\infty\end{aligned}$$

So, the answer is A.

89. Evaluate each statement.

A: Because $\lim_{x \to 1^-} f(x) = \infty$ and $\lim_{x \to 1^+} = -\infty$, $\lim_{x \to 1} f(x)$ does not exist. So, $\lim_{x \to 1} f(x) = \infty$ is false.

B: Because $\lim_{x \to 3^-} f(x) = 3$ and $\lim_{x \to 3^+} = 2$, $\lim_{x \to 3^-} f(x) > \lim_{x \to 3^+} f(x)$. So, $\lim_{x \to 3^-} f(x) < \lim_{x \to 3^+} f(x)$ is false.

C: Because $\lim_{x \to 3^-} f(x) = 3$ and $\lim_{x \to 3^+} f(x) = 2$, $\lim_{x \to 3} f(x)$ does not exist. So, $\lim_{x \to 3} f(x) = 1$ is false.

D: Because $\lim_{x \to 0^+} f(x) = 2$ and $\lim_{x \to 3^+} f(x) = 2$, $\lim_{x \to 0^+} f(x) = \lim_{x \to 3^+} f(x)$. So, $\lim_{x \to 0^+} f(x) = \lim_{x \to 3^+} f(x)$ is true.

So, the answer is D.

Section 1.6 Limits at Infinity

1. $f(x) = \dfrac{2x^2}{x^2 + 2}$

No vertical asymptotes

Horizontal asymptote: $y = 2$

Matches (f).

2. $f(x) = \dfrac{2x}{\sqrt{x^2 + 2}}$

No vertical asymptotes

Horizontal asymptotes: $y = \pm 2$

Matches (c).

3. $f(x) = \dfrac{x}{x^2 + 2}$

No vertical asymptotes

Horizontal asymptote: $y = 0$

$f(1) < 1$

Matches (d).

4. $f(x) = 2 + \dfrac{x^2}{x^4 + 1}$

No vertical asymptotes

Horizontal asymptote: $y = 2$

Matches (a).

5. $f(x) = \dfrac{4 \sin x}{x^2 + 1}$

No vertical asymptotes

Horizontal asymptote: $y = 0$

$f(1) > 1$

Matches (b).

6. $f(x) = \dfrac{2x^2 - 3x + 5}{x^2 + 1}$

No vertical asymptotes

Horizontal asymptote: $y = 2$

Matches (e).

7. $\lim_{x \to \infty} \left(12 - \dfrac{3}{x^4}\right) = 12 - 0 = 12$

8. $\lim_{x \to \infty} \left(\dfrac{100}{x^6} + e\right) = 0 + e = e$

9. $\lim_{x \to \infty} 15e^{-x} = 15(0) = 0$

10. $\lim_{x \to \infty} 7e^{-x} = 7(0) = 0$

11. $\lim_{x \to -\infty} \left(x^{-3} - 9e^x\right) = 0 - 9(0) = 0$

12. $\lim_{x \to -\infty} \left(e^x - 5x^{-2}\right) = 0 - 5(0) = 0$

13.
$$\begin{aligned}\lim_{x \to \infty} \frac{6x - 3}{3x + 2} &= \lim_{x \to \infty} \frac{6 - \frac{3}{x}}{3 + \frac{2}{x}}\\ &= \frac{6 - 0}{3 + 0}\\ &= 2\end{aligned}$$

14.
$$\begin{aligned}\lim_{x \to \infty} \frac{5 - 2x}{x + 6} &= \lim_{x \to \infty} \frac{\frac{5}{x} - 2}{1 + \frac{6}{x}}\\ &= \frac{0 - 2}{1 + 0}\\ &= -2\end{aligned}$$

15.
$$\begin{aligned}\lim_{x \to -\infty} \frac{4x^2 - 3x}{3x^2 + 3} &= \lim_{x \to -\infty} \frac{4 - \frac{3}{x}}{3 + \frac{3}{x^2}}\\ &= \frac{4 - 0}{3 + 0}\\ &= \frac{4}{3}\end{aligned}$$

16. $\lim_{x\to-\infty} \frac{5x^3 + 7}{4x^3 + x} = \lim_{x\to-\infty} \frac{5 + \frac{7}{x^3}}{4 + \frac{1}{x^2}}$

$= \frac{5 + 0}{4 + 0}$

$= \frac{5}{4}$

17. (a) $h(x) = \frac{f(x)}{x^2} = \frac{5x^3 - 3}{x^2} = 5x - \frac{3}{x^2}$

$\lim_{x\to\infty} h(x) = \infty$ (Limit does not exist.)

(b) $h(x) = \frac{f(x)}{x^3} = \frac{5x^3 - 3}{x^3} = 5 - \frac{3}{x^3}$

$\lim_{x\to\infty} h(x) = 5$

(c) $h(x) = \frac{f(x)}{x^4} = \frac{5x^3 - 3}{x^4} = \frac{5}{x} - \frac{3}{x^4}$

$\lim_{x\to\infty} h(x) = 0$

18. (a) $h(x) = \frac{f(x)}{x} = \frac{-4x^2 + 2x - 5}{x} = -4x + 2 - \frac{5}{x}$

$\lim_{x\to\infty} h(x) = -\infty$ (Limit does not exist.)

(b) $h(x) = \frac{f(x)}{x^2} = \frac{-4x^2 + 2x - 5}{x^2} = -4 + \frac{2}{x} - \frac{5}{x^2}$

$\lim_{x\to\infty} h(x) = -4$

(c) $h(x) = \frac{f(x)}{x^3} = \frac{-4x^2 + 2x - 5}{x^3} = -\frac{4}{x} + \frac{2}{x^2} - \frac{5}{x^3}$

$\lim_{x\to\infty} h(x) = 0$

19. (a) $\lim_{x\to\infty} \frac{x^2 + 2}{x^3 - 1} = 0$

(b) $\lim_{x\to\infty} \frac{x^2 + 2}{x^2 - 1} = 1$

(c) $\lim_{x\to\infty} \frac{x^2 + 2}{x - 1} = \infty$ (Limit does not exist.)

20. (a) $\lim_{x\to\infty} \frac{3 - 2x}{3x^3 - 1} = 0$

(b) $\lim_{x\to\infty} \frac{3 - 2x}{3x - 1} = -\frac{2}{3}$

(c) $\lim_{x\to\infty} \frac{3 - 2x^2}{3x - 1} = -\infty$ (Limit does not exist.)

21. (a) $\lim_{x\to\infty} \frac{5 - 2x^{3/2}}{3x^2 - 4} = 0$

(b) $\lim_{x\to\infty} \frac{5 - 2x^{3/2}}{3x^{3/2} - 4} = -\frac{2}{3}$

(c) $\lim_{x\to\infty} \frac{5 - 2x^{3/2}}{3x - 4} = -\infty$ (Limit does not exist.)

22. (a) $\lim_{x\to\infty} \frac{5x^{3/2}}{4x^2 + 1} = 0$

(b) $\lim_{x\to\infty} \frac{5x^{3/2}}{4x^{3/2} + 1} = \frac{5}{4}$

(c) $\lim_{x\to\infty} \frac{5x^{3/2}}{4\sqrt{x} + 1} = \infty$ (Limit does not exist.)

23. $\lim_{x\to\infty} \frac{2x - 1}{3x + 2} = \lim_{x\to\infty} \frac{2 - (1/x)}{3 + (2/x)} = \frac{2 - 0}{3 + 0} = \frac{2}{3}$

24. $\lim_{x\to-\infty} \frac{4x^2 + 5}{x^2 + 3} = \lim_{x\to-\infty} \frac{4 + (5/x^2)}{1 + (3/x^2)} = 4$

25. $\lim_{x\to\infty} \frac{x}{x^2 - 1} = \lim_{x\to\infty} \frac{1/x}{1 - (1/x^2)} = \frac{0}{1} = 0$

26. $\lim_{x\to\infty} \frac{5x^3 + 1}{10x^3 - 3x^2 + 7} = \lim_{x\to\infty} \frac{5 + (1/x^3)}{10 - (3/x) + (7/x^3)}$

$= \frac{5 + 0}{10 - 0} = \frac{1}{2}$

27. $\lim_{x\to-\infty} \frac{-4}{3 + 3e^{-2x}} = 0$ because $3e^{-2x} \to \infty$ as $x \to -\infty$.

28. $\lim_{x\to\infty} \frac{6}{5 + 2e^{-4x}} = \frac{6}{5}$ because $\lim_{x\to\infty} (2e^{-4x}) = 0$.

29. $\lim_{x\to-\infty} \frac{x}{\sqrt{x^2 - x}}$

$= \lim_{x\to-\infty} \frac{1}{\left(\frac{\sqrt{x^2 - x}}{-\sqrt{x^2}}\right)}$

$= \lim_{x\to-\infty} \frac{-1}{\sqrt{1 - (1/x)}}$

$= -1$, (for $x < 0$, $x = -\sqrt{x^2}$)

30. $\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}}$

$= \lim_{x\to-\infty} \frac{1}{\left(\frac{\sqrt{x^2+1}}{-\sqrt{x^2}}\right)}$

$= \lim_{x\to-\infty} \frac{-1}{\sqrt{1+(1/x^2)}}$

$= -1, \left(\text{for } x < 0, x = -\sqrt{x^2}\right)$

31. $\lim_{x\to-\infty} \frac{2x+1}{\sqrt{x^2-x}}$

$= \lim_{x\to-\infty} \frac{2+\frac{1}{x}}{\left(\frac{\sqrt{x^2-x}}{-\sqrt{x^2}}\right)}$

$= \lim_{x\to-\infty} \frac{-2-\left(\frac{1}{x}\right)}{\sqrt{1-\frac{1}{x}}}$

$= -2 \left(\text{for } x < 0, x = -\sqrt{x^2}\right)$

32. $\lim_{x\to\infty} \frac{5x^2+2}{\sqrt{x^2+3}}$

$= \lim_{x\to\infty} \frac{5x^2+2}{x\sqrt{1+(3/x^2)}}$

$= \lim_{x\to\infty} \frac{5x^2+(2/x)}{\sqrt{1+(3/x^2)}}$

$= \infty$

Limit does not exist.

33. $\lim_{x\to\infty} \frac{\sqrt{x^2-1}}{2x-1} = \lim_{x\to\infty} \frac{\sqrt{x^2-1}/\sqrt{x^2}}{2-(1/x)}$

$= \lim_{x\to\infty} \frac{\sqrt{1-(1/x^2)}}{2-(1/x)} = \frac{1}{2}$

34. $\lim_{x\to-\infty} \frac{\sqrt{x^4-1}}{x^3-1} = \lim_{x\to-\infty} \frac{\sqrt{x^4-1}}{x^3-1}\left(\frac{1/\left(-\sqrt{x^6}\right)}{1/x^3}\right)$

$= \lim_{x\to-\infty} \frac{\sqrt{(1/x^2)-(1/x^6)}}{-1+(1/x^3)} = 0$

$\left(\text{for } x < 0, -\sqrt{x^6} = x^3\right)$

35. $\lim_{x\to\infty} \frac{x+1}{(x^2+1)^{1/3}} = \lim_{x\to\infty} \frac{x+1}{(x^2+1)^{1/3}}\left(\frac{1/x^{2/3}}{1/(x^2)^{1/3}}\right)$

$= \lim_{x\to\infty} \frac{x^{1/3}+(1/x^{2/3})}{\left[1+(1/x^2)\right]^{1/3}} = \infty$

Limit does not exist.

36. $\lim_{x\to-\infty} \frac{2x}{(x^6-1)^{1/3}} = \lim_{x\to-\infty} \frac{2x}{(x^6-1)^{1/3}}\left(\frac{1/x^2}{1/(x^6)^{1/3}}\right)$

$= \lim_{x\to-\infty} \frac{2/x}{\left[1-(1/x^6)\right]^{1/3}} = 0$

37. $\lim_{x\to\infty} \frac{1}{2x+\sin x} = 0$

38. $\lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = \cos 0 = 1$

39. Because $(-1/x) \le (\sin 2x)/x \le (1/x)$ for all $x \ne 0$,

$\lim_{x\to\infty} -\frac{1}{x} \le \lim_{x\to\infty} \frac{\sin 2x}{x} \le \lim_{x\to\infty} \frac{1}{x}$

$0 \le \lim_{x\to\infty} \frac{\sin 2x}{x} \le 0$

by the Squeeze Theorem.

Therefore, $\lim_{x\to\infty} \frac{\sin 2x}{x} = 0.$

40. $\lim_{x\to\infty} \frac{x-\cos x}{x} = \lim_{x\to\infty} \left(1-\frac{\cos x}{x}\right)$

$= 1 - 0 = 1$

Note:

Because $-\frac{1}{x} \le \frac{\cos x}{x} \le \frac{1}{x}$, $\lim_{x\to\infty} \frac{\cos x}{x} = 0$ by the Squeeze Theorem.

41. $\lim_{x\to\infty} \left(2-5e^{-x}\right) = 2$

42. $\lim_{x\to\infty} \frac{8}{4-10^{-x/2}} = 2$

43. $\lim_{x\to\infty} \log_{10}\left(1+10^{-x}\right) = 0$

44. $\lim_{x\to\infty} \left(\frac{5}{2}+\ln\frac{x^2+1}{x^2}\right) = \frac{5}{2}$

45. $\lim_{t\to\infty} \left(8t^{-1}-\arctan t\right) = \lim_{t\to\infty} \left(\frac{8}{t}\right) - \lim_{t\to\infty} \arctan t$

$= 0 - \frac{\pi}{2} = -\frac{\pi}{2}$

46. $\lim_{u\to\infty} \operatorname{arcsec}(u + 1) = \dfrac{\pi}{2}$

47. The -5 in the denominator was not divided by x^3 when rewriting the function in an equivalent form.

$$\lim_{x\to-\infty} \frac{5x^3}{6x^3 - 5} = \lim_{x\to-\infty} \frac{5x^3/x^3}{6x^3/x^3 - 5/x^3}$$
$$= \lim_{x\to-\infty} \frac{5}{6 - 5/x^3}$$
$$= \frac{5}{6 - 0}$$
$$= \frac{5}{6}$$

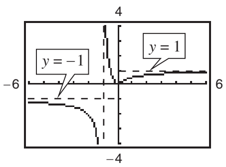
48. The denominator was not divided by $-\sqrt{x^2}$ when rewriting the function in an equivalent form.

$$\lim_{x\to-\infty} \frac{4x}{\sqrt{x^2 + 8}} = \lim_{x\to-\infty} \frac{4x/x}{\sqrt{x^2 + 8}\big/\left(-\sqrt{x^2}\right)}$$
$$= \lim_{x\to-\infty} -\frac{4}{\sqrt{1 + \dfrac{8}{x^2}}}$$
$$= -\frac{4}{\sqrt{1 + 0}}$$
$$= -4$$

49. $f(x) = \dfrac{|x|}{x + 1}$

$$\lim_{x\to\infty} \frac{|x|}{x + 1} = 1$$
$$\lim_{x\to-\infty} \frac{|x|}{x + 1} = -1$$

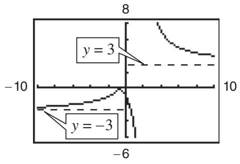
Therefore, $y = 1$ and $y = -1$ are both horizontal asymptotes.



50. $f(x) = \dfrac{|3x + 2|}{x - 2}$

$y = 3$ is a horizontal asymptote (to the right).

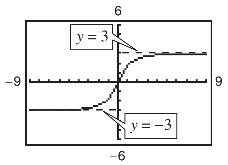
$y = -3$ is a horizontal asymptote (to the left).



51. $f(x) = \dfrac{3x}{\sqrt{x^2 + 2}}$

$$\lim_{x\to\infty} f(x) = 3$$
$$\lim_{x\to-\infty} f(x) = -3$$

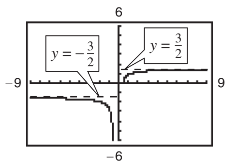
Therefore, $y = 3$ and $y = -3$ are both horizontal asymptotes.



52. $f(x) = \dfrac{\sqrt{9x^2 - 2}}{2x + 1}$

$y = \dfrac{3}{2}$ is a horizontal asymptote (to the right).

$y = -\dfrac{3}{2}$ is a horizontal asymptote (to the left).



53. $\lim_{t\to\infty} N(t) = \infty$

$\lim_{t\to\infty} E(t) = c$

54. (a) The function is even: $\lim_{x\to-\infty} f(x) = 5$.

(b) The function is odd: $\lim_{x\to-\infty} f(x) = -5$.

55. $\lim_{v_1/v_2\to\infty} 100\left[1 - \dfrac{1}{(v_1/v_2)^c}\right] = 100[1 - 0] = 100\%$

56. $C = 0.5x + 500$

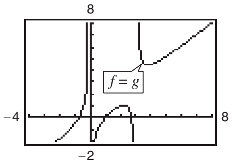
$\overline{C} = \dfrac{C}{x}$

$\overline{C} = 0.5 + \dfrac{500}{x}$

$\lim_{x\to\infty}\left(0.5 + \dfrac{500}{x}\right) = 0.5$

57. $f(x) = \dfrac{x^3 - 3x^2 + 2}{x(x-3)}$, $g(x) = x + \dfrac{2}{x(x-3)}$

(a)

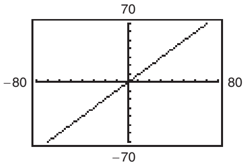


(b) $f(x) = \dfrac{x^3 - 3x^2 + 2}{x(x-3)}$

$= \dfrac{x^2(x-3)}{x(x-3)} + \dfrac{2}{x(x-3)}$

$= x + \dfrac{2}{x(x-3)} = g(x)$

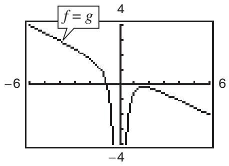
(c)



The graph appears as the slant asymptote $y = x$.

58. $f(x) = -\dfrac{x^3 - 2x^2 + 2}{2x^2}$, $g(x) = -\dfrac{1}{2}x + 1 - \dfrac{1}{x^2}$

(a)

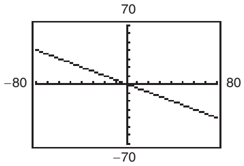


(b) $f(x) = -\dfrac{x^3 - 2x^2 + 2}{2x^2}$

$= -\left[\dfrac{x^3}{2x^2} - \dfrac{2x^2}{2x^2} + \dfrac{2}{2x^2}\right]$

$= -\dfrac{1}{2}x + 1 - \dfrac{1}{x^2} = g(x)$

(c)



The graph appears as the slant asymptote $y = -\dfrac{1}{2}x + 1$.

59. $\lim_{x\to\infty} x\sin\dfrac{1}{x} = \lim_{t\to 0^+}\dfrac{\sin t}{t} = 1$

(Let $x = 1/t$.)

60. $\lim_{x\to\infty} x\tan\dfrac{1}{x} = \lim_{x\to 0^+}\dfrac{\tan t}{t} = \lim_{x\to 0^+}\left[\dfrac{\sin t}{t}\cdot\dfrac{1}{\cos t}\right]$

$= (1)(1) = 1$

(Let $x = 1/t$.)

61. $\lim_{x\to-\infty}\left(x + \sqrt{x^2+3}\right) = \lim_{x\to-\infty}\left[\left(x + \sqrt{x^2+3}\right)\cdot\dfrac{x - \sqrt{x^2+3}}{x - \sqrt{x^2+3}}\right] = \lim_{x\to-\infty}\dfrac{-3}{x - \sqrt{x^2+3}} = 0$

62. $\lim_{x\to\infty}\left(x - \sqrt{x^2+x}\right) = \lim_{x\to\infty}\left[\left(x - \sqrt{x^2+x}\right)\cdot\dfrac{x + \sqrt{x^2+x}}{x + \sqrt{x^2+x}}\right] = \lim_{x\to\infty}\dfrac{-x}{x + \sqrt{x^2+x}} = \lim_{x\to\infty}\dfrac{-1}{1 + \sqrt{1 + (1/x)}} = -\dfrac{1}{2}$

63. $\lim_{x\to-\infty}\left(3x + \sqrt{9x^2 - x}\right) = \lim_{x\to-\infty}\left[\left(3x + \sqrt{9x^2 - x}\right)\cdot\dfrac{3x - \sqrt{9x^2 - x}}{3x - \sqrt{9x^2 - x}}\right]$

$= \lim_{x\to-\infty}\dfrac{x}{3x - \sqrt{9x^2 - x}}$

$= \lim_{x\to-\infty}\dfrac{1}{3 - \dfrac{\sqrt{9x^2 - x}}{-\sqrt{x^2}}}\left(\text{for } x < 0,\ x = -\sqrt{x^2}\right)$

$= \lim_{x\to-\infty}\dfrac{1}{3 + \sqrt{9 - (1/x)}} = \dfrac{1}{6}$

64. $\lim_{x\to\infty}\left(4x - \sqrt{16x^2 - x}\right)\dfrac{4x + \sqrt{16x^2 - x}}{4x + \sqrt{16x^2 - x}} = \lim_{x\to\infty}\dfrac{16x^2 - \left(16x^2 - x\right)}{4x + \sqrt{\left(16x^2 - x\right)}}$

$$= \lim_{x\to\infty}\frac{x}{4x + \sqrt{16x^2 - x}}$$

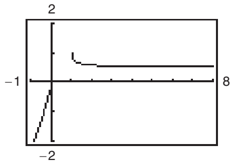
$$= \lim_{x\to\infty}\frac{1}{4 + \sqrt{16 - 1/x}}$$

$$= \frac{1}{4 + 4} = \frac{1}{8}$$

65.

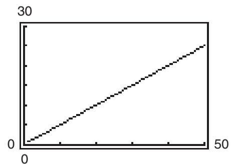
x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	1	0.513	0.501	0.500	0.500	0.500	0.500

$$\lim_{x\to\infty}\left[x - \sqrt{x(x-1)}\right] = \lim_{x\to\infty}\frac{x - \sqrt{x^2 - x}}{1}\cdot\frac{x + \sqrt{x^2 - x}}{x + \sqrt{x^2 - x}} = \lim_{x\to\infty}\frac{x}{x + \sqrt{x^2 - x}} = \lim_{x\to\infty}\frac{1}{1 + \sqrt{1 - (1/x)}} = \frac{1}{2}$$



66.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	1.0	5.1	50.1	500.1	5000.1	50,000.1	500,000.1

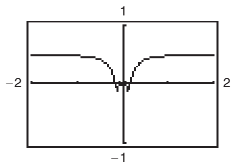


$$\lim_{x\to\infty}\frac{x^2 - x\sqrt{x^2 - x}}{1}\cdot\frac{x^2 + x\sqrt{x^2 - x}}{x^2 + x\sqrt{x^2 - x}} = \lim_{x\to\infty}\frac{x^3}{x^2 + x\sqrt{x^2 - x}} = \infty$$

Limit does not exist.

67.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	0.479	0.500	0.500	0.500	0.500	0.500	0.500

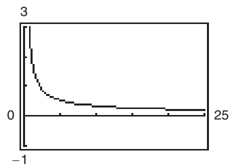


Let $x = 1/t$.

$$\lim_{x\to\infty} x\sin\left(\frac{1}{2x}\right) = \lim_{t\to 0^+}\frac{\sin(t/2)}{t} = \lim_{t\to 0^+}\frac{1}{2}\frac{\sin(t/2)}{t/2} = \frac{1}{2}$$

68.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	2.000	0.348	0.101	0.032	0.010	0.003	0.001



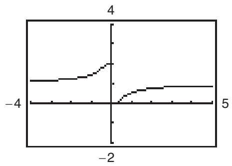
$$\lim_{x\to\infty}\frac{x + 1}{x\sqrt{x}} = 0$$

69. (a) $\lim_{x\to\infty} f(x) = 4$ means that $f(x)$ approaches 4 as x becomes large.

(b) $\lim_{x\to-\infty} f(x) = 2$ means that $f(x)$ approaches 2 as x becomes very large (in absolute value) and negative.

70. Answers will vary.

71. (a)



(b) Answers will vary. Sample answer: When x increases without bound, $1/x$ approaches zero and $e^{1/x}$ approaches 1. Therefore, $f(x)$ approaches $2/(1 + 1) = 1$. So, $f(x)$ has a horizontal asymptote at $y = 1$. As x approaches zero from the right, $1/x$ approaches ∞, $e^{1/x}$ approaches ∞, and $f(x)$ approaches zero. As x approaches zero from the left, $1/x$ approaches $-\infty$, $e^{1/x}$ approaches zero, and $f(x)$ approaches 2. The limit does not exist because the left limit does not equal the right limit. Therefore, $x = 0$ is a nonremovable discontinuity.

72. (a) $\lim_{t \to 0^+} T = 1700°$

This is the temperature of the kiln.

(b) $\lim_{t \to \infty} T = 72°$

This is the temperature of the room.

73. $f(x) = \dfrac{2x^2}{x^2 + 2}$

(a) $\lim_{x \to \infty} f(x) = 2 = L$

$\lim_{x \to \infty} f(x) = 2 = K$

(b) $f(x_1) + \varepsilon = \dfrac{2x_1^2}{x_1^2 + 2} + \varepsilon = 2$

$$\begin{aligned} 2x_1^2 + \varepsilon x_1^2 + 2\varepsilon &= 2x_1^2 + 4 \\ x_1^2\varepsilon &= 4 - 2\varepsilon \\ x_1 &= \sqrt{\frac{4 - 2\varepsilon}{\varepsilon}} \\ x_2 &= -x_1 \text{ by symmetry} \end{aligned}$$

(c) $M = x_1 = \sqrt{\dfrac{4 - 2\varepsilon}{\varepsilon}}$

(d) $N = x_2 = -\sqrt{\dfrac{4 - 2\varepsilon}{\varepsilon}}$

74. $f(x) = \dfrac{6x}{\sqrt{x^2 + 2}}$

(a) $\lim_{x \to \infty} f(x) = 6 = L$

$\lim_{x \to -\infty} f(x) = -6 = K$

(b) $f(x_1) + \varepsilon = \dfrac{6x_1}{\sqrt{x_1^2 + 2}} + \varepsilon = 6$

$$\begin{aligned} 6x_1 &= (6 - \varepsilon)\sqrt{x_1^2 + 2} \\ 36x_1^2 &= (x_1^2 + 2)(6 - \varepsilon)^2 \\ 36x_1^2 - (6 - \varepsilon)^2 x_1^2 &= 2(6 - \varepsilon)^2 \\ x_1^2\left[36 - 36 + 12\varepsilon - \varepsilon^2\right] &= 2(6 - \varepsilon)^2 \\ x_1^2 &= \frac{2(6 - \varepsilon)^2}{12\varepsilon - \varepsilon^2} \\ x_1 &= (6 - \varepsilon)\sqrt{\frac{2}{12\varepsilon - \varepsilon^2}} \\ x_2 &= -x_1 \text{ by symmetry} \end{aligned}$$

(c) $M = x_1 = (6 - \varepsilon)\sqrt{\dfrac{2}{12\varepsilon - \varepsilon^2}}$

(d) $N = x_2 = (\varepsilon - 6)\sqrt{\dfrac{2}{12\varepsilon - \varepsilon^2}}$

75. $\lim_{x \to \infty} \dfrac{3x}{\sqrt{x^2 + 3}} = 3$

$f(x_1) + \varepsilon = \dfrac{3x_1}{\sqrt{x_1^2 + 3}} + \varepsilon = 3$

$$\begin{aligned} 3x_1 &= (3 - \varepsilon)\sqrt{x_1^2 + 3} \\ 9x_1^2 &= (3 - \varepsilon)^2(x_1^2 + 3) \\ 9x_1^2 - (3 - \varepsilon)^2 x_1^2 &= 3(3 - \varepsilon)^2 \\ x_1^2(9 - 9 + 6\varepsilon - \varepsilon^2) &= 3(3 - \varepsilon)^2 \\ x_1^2 &= \frac{3(3 - \varepsilon)^2}{6\varepsilon - \varepsilon^2} \\ x_1 &= (3 - \varepsilon)\sqrt{\frac{3}{6\varepsilon - \varepsilon^2}} \end{aligned}$$

Let $M = x_1 = (3 - \varepsilon)\sqrt{\dfrac{3}{6\varepsilon - \varepsilon^2}}$

(a) When $\varepsilon = 0.5$:

$$M = (3 - 0.5)\sqrt{\frac{3}{6(0.5) - (0.5)^2}} = \frac{5\sqrt{33}}{11}$$

(b) When $\varepsilon = 0.1$:

$$M = (3 - 0.1)\sqrt{\frac{3}{6(0.1) - (0.1)^2}} = \frac{29\sqrt{177}}{59}$$

76. Yes. For example, $f(x) = \dfrac{\sin x}{x}$ crosses $y = 0$.

77. $\lim_{x\to\infty} \dfrac{1}{x^2} = 0$. Let $\varepsilon > 0$ be given. You need $M > 0$ such that

$$\left|f(x) - L\right| = \left|\frac{1}{x^2} - 0\right| = \frac{1}{x^2} < \varepsilon \text{ whenever } x > M.$$

$$x^2 > \frac{1}{\varepsilon} \Rightarrow x > \frac{1}{\sqrt{\varepsilon}}$$

Let $M = \dfrac{1}{\sqrt{\varepsilon}}$.

For $x > M$, you have

$$x > \frac{1}{\sqrt{\varepsilon}} \Rightarrow x^2 > \frac{1}{\varepsilon} \Rightarrow \frac{1}{x^2} < \varepsilon \Rightarrow \left|f(x) - L\right| < \varepsilon.$$

78. $\lim_{x\to\infty} \dfrac{2}{\sqrt{x}} = 0$. Let $\varepsilon > 0$ be given. You need $M > 0$ such that

$$\left|f(x) - L\right| = \left|\frac{2}{\sqrt{x}} - 0\right| = \frac{2}{\sqrt{x}} < \varepsilon \text{ whenever}$$

$x > M$.

$$\frac{2}{\sqrt{x}} < \varepsilon \Rightarrow \frac{\sqrt{x}}{2} > \frac{1}{\varepsilon} \Rightarrow x > \frac{4}{\varepsilon^2}$$

Let $M = 4/\varepsilon^2$.

For $x > M = 4/\varepsilon^2$, you have

$$\sqrt{x} > 2/\varepsilon \Rightarrow \frac{2}{\sqrt{x}} < \varepsilon \Rightarrow \left|f(x) - L\right| < \varepsilon.$$

79. $\lim_{x\to-\infty} \dfrac{1}{x^3} = 0$. Let $\varepsilon > 0$. You need $N < 0$ such that

$$\left|f(x) - L\right| = \left|\frac{1}{x^3} - 0\right| = \frac{-1}{x^3} < \varepsilon \text{ whenever } x < N.$$

$$\frac{-1}{x^3} < \varepsilon \Rightarrow -x^3 > \frac{1}{\varepsilon} \Rightarrow x < \frac{-1}{\varepsilon^{1/3}}$$

Let $N = \dfrac{-1}{\sqrt[3]{\varepsilon}}$.

For $x < N = \dfrac{-1}{\sqrt[3]{\varepsilon}}$,

$$\frac{1}{x} > -\sqrt[3]{\varepsilon}$$

$$-\frac{1}{x} < \sqrt[3]{\varepsilon}$$

$$-\frac{1}{x^3} < \varepsilon$$

$$\Rightarrow \left|f(x) - L\right| < \varepsilon.$$

80. $\lim_{x\to-\infty} \dfrac{1}{x-2} = 0$. Let $\varepsilon > 0$ be given.

You need $N < 0$ such that

$$\left|f(x) - L\right| = \left|\frac{1}{x-2} - 0\right| = \frac{-1}{x-2} < \varepsilon$$

whenever $x < N$.

$$\frac{-1}{x-2} < \varepsilon \Rightarrow x - 2 < \frac{-1}{\varepsilon} \Rightarrow x < 2 - \frac{1}{\varepsilon}$$

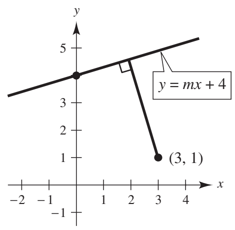
Let $N = 2 - \dfrac{1}{\varepsilon}$. For $x < N = 2 - \dfrac{1}{\varepsilon}$,

$$x - 2 < \frac{-1}{\varepsilon}$$

$$\frac{-1}{x-2} < \varepsilon$$

$$\Rightarrow \left|f(x) - L\right| < \varepsilon.$$

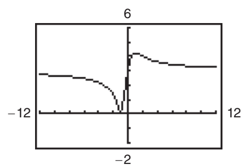
81. Line: $y = mx + 4$



(a) $d = \dfrac{\left|Ax_1 + By_1 + C\right|}{\sqrt{A^2 + B^2}} = \dfrac{\left|m(3) - 1(1) + 4\right|}{\sqrt{m^2 + 1}}$

$= \dfrac{\left|3m + 3\right|}{\sqrt{m^2 + 1}}$

(b)



(c) $\lim_{m\to\infty} d(m) = 3 = \lim_{m\to\infty} d(m)$

The line approaches the vertical line $x = 0$. So, the distance from $(3, 1)$ approaches 3.

82. $\lim_{x\to\infty} x^3 = \infty$. Let $M > 0$ be given. You need $N > 0$ such that $f(x) = x^3 > M$ whenever $x > N$.

$x^3 > M \Rightarrow x > M^{1/3}$. Let $N = M^{1/3}$. For $x > N = M^{1/3}$, $x > M^{1/3} \Rightarrow x^3 > M \Rightarrow f(x) > M$.

83. Evaluate each statement.

A: $f(10)$ may or may not be undefined based on the function f.
The statement may or may not be true.

B: $\lim_{x\to 10} f(x)$ may or may not exist based on the function f.
The statement may or may not be true.

C: Because $y = 10$ is a horizontal asymptote, $\lim_{x\to\infty} f(x) = 10$.
The statement must be true.

D: Even though $y = 10$ is a horizontal asymptote, there may be at least one value of x for which $f(x) = 10$.
The statement may or may not be true.

So, the answer is C.

84. $$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^3 + x - 5}{4x^2 + 8 - 5x^3} = \lim_{x\to\infty} \frac{\dfrac{x^3}{x^3} + \dfrac{x}{x^3} - \dfrac{5}{x^3}}{\dfrac{4x^2}{x^3} + \dfrac{8}{x^3} - \dfrac{5x^3}{x^3}} = \lim_{x\to\infty} \frac{1 + \dfrac{1}{x^2} - \dfrac{5}{x^3}}{\dfrac{4}{x} + \dfrac{8}{x^3} - 5} = \frac{1 + 0 - 0}{0 + 0 - 5} = -\frac{1}{5}$$

Because $\lim_{x\to\infty} f(x) = -\frac{1}{5}$, the graph has a horizontal asymptote at $y = -\frac{1}{5}$. So, the answer is B.

85. $$\lim_{x\to\infty} \frac{\sqrt{16x^2 - 5}}{6x - 1} = \lim_{x\to\infty} \frac{\dfrac{\sqrt{16x^2 - 5}}{\sqrt{x^2}}}{\dfrac{6x - 1}{x}} = \lim_{x\to\infty} \frac{\sqrt{16 - \dfrac{5}{x^2}}}{6 - \dfrac{1}{x}} = \frac{\sqrt{16 - 0}}{6 - 0} = \frac{4}{6} = \frac{2}{3}$$

So, the answer is A.

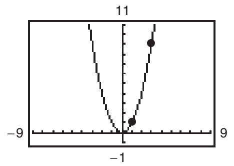
86. From the table, the student's average typing speed appears to be approaching 100 words per minute.

$$\lim_{t\to\infty} \frac{100t^2}{65 + t^2} = \lim_{t\to\infty} \frac{100}{\dfrac{65}{t^2} + 1} = \frac{100}{0 + 1} = 100$$

So, the answer is D.

Review Exercises for Chapter 1

1. Calculus required. Using a graphing utility, you can estimate the length to be 8.3. Or, the length is slightly longer than the distance between the two points, approximately 8.25.

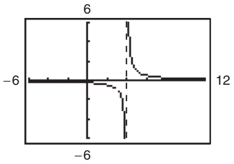


2. Precalculus. $L = \sqrt{(9 - 1)^2 + (3 - 1)^2} \approx 8.25$

3. $f(x) = \dfrac{x-3}{x^2 - 7x + 12}$

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$	–0.9091	–0.9901	–0.9990	?	–1.0010	–1.0101	–1.1111

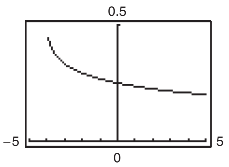
$\lim_{x\to 3} f(x) \approx -1.0000$ (Actual limit is -1.)



4. $f(x) = \dfrac{\sqrt{x+4} - 2}{x}$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	0.2516	0.2502	0.2500	?	0.2500	0.2498	0.2485

$\lim_{x\to 0} f(x) \approx 0.2500$ $\left(\text{Actual limit is } \tfrac{1}{4}.\right)$



5. $h(x) = \dfrac{4x - x^2}{x} = \dfrac{x(4-x)}{x} = 4 - x, x \neq 0$

(a) $\lim_{x\to 0} h(x) = 4 - 0 = 4$

(b) $\lim_{x\to -1} h(x) = 4 - (-1) = 5$

6. $f(t) = \dfrac{\ln(t+2)}{t}$

(a) $\lim_{t\to 0} f(t)$ does not exist because $\lim_{t\to 0^-} f(t) = -\infty$ and $\lim_{t\to 0^+} f(t) = \infty$.

(b) $\lim_{t\to -1} f(t) = \dfrac{\ln 1}{-1} = 0$

7. $\lim_{x\to 1} (x + 4) = 1 + 4 = 5$

Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon$. Then for $0 < |x - 1| < \delta = \varepsilon$, you have

$$|x - 1| < \varepsilon$$
$$|(x + 4) - 5| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

8. $\lim_{x \to 9} \sqrt{x} = \sqrt{9} = 3$

Let $\varepsilon > 0$ be given. You need

$\left|\sqrt{x} - 3\right| < \varepsilon \Rightarrow \left|\sqrt{x} + 3\right|\left|\sqrt{x} - 3\right| < \varepsilon\left|\sqrt{x} + 3\right| \Rightarrow \left|x - 9\right| < \varepsilon\left|\sqrt{x} + 3\right|.$

Assuming $4 < x < 16$, you can choose $\delta = 5\varepsilon$.

So, for $0 < \left|x - 9\right| < \delta = 5\varepsilon$, you have

$$\begin{aligned} \left|x - 9\right| &< 5\varepsilon < \left|\sqrt{x} + 3\right|\varepsilon \\ \left|\sqrt{x} - 3\right| &< \varepsilon \\ \left|f(x) - L\right| &< \varepsilon. \end{aligned}$$

9. $\lim_{x \to 2} \left(1 - x^2\right) = 1 - 2^2 = -3$

Let $\varepsilon > 0$ be given. You need

$\left|1 - x^2 - (-3)\right| < \varepsilon \Rightarrow \left|x^2 - 4\right| = \left|x - 2\right|\left|x + 2\right| < \varepsilon \Rightarrow \left|x - 2\right| < \dfrac{1}{\left|x + 2\right|}\varepsilon$

Assuming $1 < x < 3$, you can choose $\delta = \dfrac{\varepsilon}{5}$.

So, for $0 < \left|x - 2\right| < \delta = \dfrac{\varepsilon}{5}$, you have

$$\begin{aligned} \left|x - 2\right| &< \frac{\varepsilon}{5} < \frac{\varepsilon}{\left|x + 2\right|} \\ \left|x - 2\right|\left|x + 2\right| &< \varepsilon \\ \left|x^2 - 4\right| &< \varepsilon \\ \left|4 - x^2\right| &< \varepsilon \\ \left|\left(1 - x^2\right) - (-3)\right| &< \varepsilon \\ \left|f(x) - L\right| &< \varepsilon. \end{aligned}$$

10. $\lim_{x \to 5} 9 = 9$. Let $\varepsilon > 0$ be given. δ can be any positive number. So, for $0 < \left|x - 5\right| < \delta$, you have

$$\begin{aligned} \left|9 - 9\right| &< \varepsilon \\ \left|f(x) - L\right| &< \varepsilon. \end{aligned}$$

11. $\lim_{x \to -6} x^2 = (-6)^2 = 36$

12. $\lim_{x \to 0} \left(3x - 5\right) = 3(0) - 5 = -5$

13. $\lim_{x \to -5} \sqrt[3]{x - 3} = \sqrt[3]{(-5) - 3} = \sqrt[3]{-8} = -2$

14. $\lim_{x \to 6} \left(x - 2\right)^2 = \left(6 - 2\right)^2 = 16$

15. $\lim_{x \to 3} \dfrac{3}{x - 1} = \dfrac{3}{3 - 1} = \dfrac{3}{2}$

16. $\lim_{x \to 2} \dfrac{x}{x^2 + 1} = \dfrac{2}{2^2 + 1} = \dfrac{2}{4 + 1} = \dfrac{2}{5}$

17. $\lim_{t \to -2} \dfrac{t + 2}{t^2 - 4} = \lim_{t \to -2} \dfrac{1}{t - 2} = -\dfrac{1}{4}$

18. $\lim_{t \to 4} \dfrac{t^2 - 16}{t - 4} = \lim_{t \to 4} \dfrac{(t - 4)(t + 4)}{t - 4}$

$= \lim_{t \to 4} (t + 4) = 4 + 4 = 8$

19. $$\lim_{x\to5}\frac{\sqrt{x-4}-1}{x-5} = \lim_{x\to5}\frac{\left(\sqrt{x-4}-1\right)\left(\sqrt{x-4}+1\right)}{(x-5)\left(\sqrt{x-4}+1\right)}$$
$$= \lim_{x\to5}\frac{(x-4)-1}{(x-5)\left(\sqrt{x-4}+1\right)}$$
$$= \lim_{x\to5}\frac{x-5}{(x-5)\left(\sqrt{x-4}+1\right)}$$
$$= \lim_{x\to5}\frac{1}{\sqrt{x-4}+1}$$
$$= \frac{1}{\sqrt{5-4}+1}$$
$$= \frac{1}{2}$$

20. $$\lim_{x\to0}\frac{\sqrt{1+x}-1}{3x} = \lim_{x\to0}\frac{\left(\sqrt{1+x}-1\right)\left(\sqrt{1+x}+1\right)}{3x\left(\sqrt{1+x}+1\right)}$$
$$= \lim_{x\to0}\frac{(1+x)-1}{3x\left(\sqrt{1+x}+1\right)}$$
$$= \lim_{x\to0}\frac{x}{3x\left(\sqrt{1+x}+1\right)}$$
$$= \lim_{x\to0}\frac{1}{3\left(\sqrt{1+x}+1\right)}$$
$$= \frac{1}{3\left(\sqrt{1+0}+1\right)}$$
$$= \frac{1}{6}$$

21. $$\lim_{x\to0}\frac{[1/(x+1)]-1}{x} = \lim_{x\to0}\frac{1-(x+1)}{x(x+1)} = \lim_{x\to0}\frac{-1}{x+1} = -1$$

22. $$\lim_{s\to0}\frac{\left(1/\sqrt{1+s}\right)-1}{s} = \lim_{s\to0}\left[\frac{\left(1/\sqrt{1+s}\right)-1}{s}\cdot\frac{\left(1/\sqrt{1+s}\right)+1}{\left(1/\sqrt{1+s}\right)+1}\right]$$
$$= \lim_{s\to0}\frac{[1/(1+s)]-1}{s\left[\left(1/\sqrt{1+s}\right)+1\right]} = \lim_{s\to0}\frac{-1}{(1+s)\left[\left(1/\sqrt{1+s}\right)+1\right]} = -\frac{1}{2}$$

23. $$\lim_{x\to0}\frac{1-\cos x}{\sin x} = \lim_{x\to0}\left(\frac{x}{\sin x}\right)\left(\frac{1-\cos x}{x}\right) = (1)(0) = 0$$

24. $$\lim_{x\to(\pi/4)}\frac{4x}{\tan x} = \frac{4(\pi/4)}{1} = \pi$$

25. $$\lim_{x\to1} e^{x-1}\sin\frac{\pi x}{2} = e^0\sin\frac{\pi}{2} = 1$$

26. $$\lim_{x\to2}\frac{\ln(x-1)^2}{\ln(x-1)} = \lim_{x\to2}\frac{2\ln(x-1)}{\ln(x-1)} = \lim_{x\to2} 2 = 2$$

27. $$\lim_{\Delta x\to0}\frac{\sin[(\pi/6)+\Delta x]-(1/2)}{\Delta x} = \lim_{\Delta x\to0}\frac{\sin(\pi/6)\cos\Delta x+\cos(\pi/6)\sin\Delta x-(1/2)}{\Delta x}$$
$$= \lim_{\Delta x\to0}\frac{1}{2}\cdot\frac{(\cos\Delta x-1)}{\Delta x} + \lim_{\Delta x\to0}\frac{\sqrt{3}}{2}\cdot\frac{\sin\Delta x}{\Delta x} = 0 + \frac{\sqrt{3}}{2}(1) = \frac{\sqrt{3}}{2}$$

28. $$\lim_{\Delta x\to0}\frac{\cos(\pi+\Delta x)+1}{\Delta x} = \lim_{\Delta x\to0}\frac{\cos\pi\cos\Delta x-\sin\pi\sin\Delta x+1}{\Delta x}$$
$$= \lim_{\Delta x\to0}\left[-\frac{(\cos\Delta x-1)}{\Delta x}\right] - \lim_{\Delta x\to0}\left[\sin\pi\frac{\sin\Delta x}{\Delta x}\right]$$
$$= -0-(0)(1) = 0$$

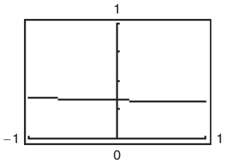
29. $$\lim_{x\to c}[f(x)g(x)] = \left[\lim_{x\to c}f(x)\right]\left[\lim_{x\to c}g(x)\right]$$
$$= (-6)\left(\tfrac{1}{2}\right) = -3$$

30. $$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c}f(x)}{\lim_{x\to c}g(x)} = \frac{-6}{\left(\frac{1}{2}\right)} = -12$$

31. $$\lim_{x\to c}[f(x)+2g(x)] = \lim_{x\to c}f(x) + 2\lim_{x\to c}g(x)$$
$$= -6 + 2\left(\tfrac{1}{2}\right) = -5$$

32. $$\lim_{x\to c}[f(x)]^2 = \left[\lim_{x\to c}f(x)\right]^2$$
$$= (-6)^2 = 36$$

33. $f(x) = \dfrac{\sqrt{2x+9} - 3}{x}$



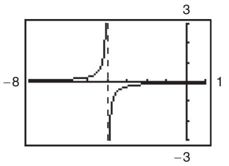
The limit appears to be $\frac{1}{3}$.

x	–0.01	–0.001	0	0.001	0.01
$f(x)$	0.3335	0.3333	?	0.3333	0.331

$\lim_{x \to 0} f(x) \approx 0.3333$

$$\lim_{x \to 0} \frac{\sqrt{2x+9} - 3}{x} \cdot \frac{\sqrt{2x+9} + 3}{\sqrt{2x+9} + 3} = \lim_{x \to 0} \frac{(2x+9) - 9}{x\left[\sqrt{2x+9} + 3\right]} = \lim_{x \to 0} \frac{2}{\sqrt{2x+9} + 3} = \frac{2}{\sqrt{9} + 3} = \frac{1}{3}$$

34. $f(x) = \dfrac{[1/(x+4)] - (1/4)}{x}$



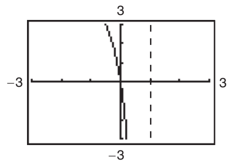
The limit appears to be $-\frac{1}{16}$.

x	–0.01	–0.001	0	0.001	0.01
$f(x)$	–0.0627	–0.0625	?	–0.0625	–0.0623

$\lim_{x \to 0} f(x) \approx -0.0625 = -\frac{1}{16}$

$$\lim_{x \to 0} \frac{\frac{1}{x+4} - \frac{1}{4}}{x} = \lim_{x \to 0} \frac{4 - (x+4)}{(x+4)4(x)} = \lim_{x \to 0} \frac{-1}{(x+4)4} = -\frac{1}{16}$$

35. $f(x) = \lim_{x \to 0} \dfrac{20(e^{x/2} - 1)}{x - 1}$



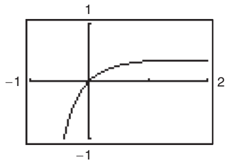
The limit appears to be 0.

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	0.8867	0.0988	0.0100	?	–0.0100	–0.1013	–1.1394

$\lim_{x \to 0} f(x) \approx 0.0000$

$$\lim_{x \to 0} \frac{20(e^{x/2} - 1)}{x - 1} = \frac{20(e^0 - 1)}{0 - 1} = \frac{0}{-1} = 0$$

36. $f(x) = \dfrac{\ln(x+1)}{x+1}$



The limit appears to be 0.

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	–0.1171	–0.0102	–0.0010	?	0.0010	0.0099	0.0866

$\lim_{x\to 0} f(x) \approx 0.0000$

$$\lim_{x\to 0}\frac{\ln(x+1)}{x+1} = \frac{\ln 1}{1} = \frac{0}{1} = 0$$

37.
$$\begin{aligned} v &= \lim_{t\to 4}\frac{s(4)-s(t)}{4-t} \\ &= \lim_{t\to 4}\frac{\left[-4.9(16)+250\right]-\left[-4.9t^2+250\right]}{4-t} \\ &= \lim_{t\to 4}\frac{4.9(t^2-16)}{4-t} \\ &= \lim_{t\to 4}\frac{4.9(t-4)(t+4)}{4-t} \\ &= \lim_{t\to 4}\left[-4.9(t+4)\right] = -39.2 \text{ m/sec} \end{aligned}$$

The object is falling at about 39.2 m/sec.

38.
$$\begin{aligned} -4.9t^2 + 250 &= 0 \\ 250 &= 4.9t^2 \\ \frac{250}{4.9} &= t^2 \\ \frac{50}{7} &= t \end{aligned}$$

The object will hit the ground after $t = \frac{50}{7}$ seconds.

When $a = \frac{50}{7}$, the velocity is

$$\begin{aligned} \lim_{t\to a}\frac{s(a)-s(t)}{a-t} &= \lim_{t\to a}\frac{\left[-4.9a^2+250\right]-\left[-4.9t^2+250\right]}{a-t} \\ &= \lim_{t\to a}\frac{4.9(t^2-a^2)}{a-t} \\ &= \lim_{t\to a}\frac{4.9(t-a)(t+a)}{a-t} \\ &= \lim_{t\to a}\left[-4.9(t+a)\right] \\ &= -4.9(2a) \qquad \left(a = \frac{50}{7}\right) \\ &= -70 \text{ m/sec.} \end{aligned}$$

The velocity of the object when it hits the ground is about 70 m/sec.

39. $f(x) = \sqrt{(x-1)x}$

(a) Domain: $(-\infty, 0] \cup [1, \infty)$

(b) $\lim\limits_{x\to 0^-} f(x) = 0$

(c) $\lim\limits_{x\to 1^+} f(x) = 0$

40. $f(x) = \dfrac{x^2-4}{|x-2|} = (x+2)\left[\dfrac{x-2}{|x-2|}\right]$

(a) $\lim\limits_{x\to 2^-} f(x) = -4$

(b) $\lim\limits_{x\to 2^+} f(x) = 4$

(c) $\lim\limits_{x\to 2} f(x)$ does not exist.

41. $\lim\limits_{x\to 6^+} \dfrac{1}{x+6} = \dfrac{1}{6+6} = \dfrac{1}{12}$

42.
$$\lim_{x\to 7^-} \frac{x-7}{x^2-49} = \lim_{x\to 7^-} \frac{x-7}{(x-7)(x+7)} = \lim_{x\to 7^-} \frac{1}{x+7} = \frac{1}{7+7} = \frac{1}{14}$$

43.
$$\lim_{x\to 9^-} \frac{\sqrt{x}-3}{x-9} = \lim_{x\to 9^-} \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)} = \lim_{x\to 9^-} \frac{x-9}{(x-9)(\sqrt{x}+3)} = \lim_{x\to 9^-} \frac{1}{\sqrt{x}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$$

(or: $= \lim\limits_{x\to 9^-} \dfrac{\sqrt{x}-3}{(\sqrt{x}-3)(\sqrt{x}+3)} = \lim\limits_{x\to 9^-} \dfrac{1}{\sqrt{x}+3} = \dfrac{1}{\sqrt{9}+3} = \dfrac{1}{6}$)

44. $\lim\limits_{x\to -11^-} \dfrac{|x+11|}{x+11} = \lim\limits_{x\to -11^-} \dfrac{-(x+11)}{x+11} = \lim\limits_{x\to -11^-} -1 = -1$

45. $\lim\limits_{x\to 2^-} (2[\![x]\!] + 1) = 2(1) + 1 = 3$

46. $\lim\limits_{x\to 4} [\![x-1]\!]$ does not exist. There is a break in the graph at $x = 4$.

47. $\lim\limits_{x\to 2^-} f(x) = 0$

48. $\lim\limits_{s\to -2} f(s) = 2$

49. $f(x) = x^2 - 64$ is continuous for all real x.

50. $f(x) = \dfrac{1}{x^2-9} = \dfrac{1}{(x-3)(x+3)}$

has nonremovable discontinuities at $x = \pm 3$ because $\lim\limits_{x\to 3} f(x)$ and $\lim\limits_{x\to -3} f(x)$ do not exist.

51. $f(x) = \dfrac{x}{x^3-x} = \dfrac{x}{x(x^2-1)} = \dfrac{1}{(x-1)(x+1)}, x \neq 0$

has nonremovable discontinuities at $x = \pm 1$ because $\lim\limits_{x\to -1} f(x)$ and $\lim\limits_{x\to 1} f(x)$ do not exist, and has a removable discontinuity at $x = 0$ because

$\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{1}{(x-1)(x+1)} = -1.$

52. $f(x) = \dfrac{x+6}{x^2-3x-54} = \dfrac{x+6}{(x-9)(x+6)} = \dfrac{1}{x-9},$

$x \neq -6$ has a removable discontinuity at $x = -6$ because $\lim\limits_{x\to -6} f(x) = \lim\limits_{x\to -6} \dfrac{1}{x-9} = -\dfrac{1}{15}$ and has a nonremovable discontinuity at $x = 9$ because $\lim\limits_{x\to 9} f(x)$ does not exist.

53. $f(x) = -7x^2 + 3$

f is continuous on $(-\infty, \infty)$.

54. $f(x) = \dfrac{4x^2+7x-2}{x+2} = \dfrac{(4x-1)(x+2)}{x+2}$

f is continuous on $(-\infty, -2) \cup (-2, \infty)$. There is a removable discontinuity at $x = -2$.

55. $f(x) = \sqrt{x-4}$

f is continuous on $[4, \infty)$.

56. $f(x) = [\![x + 3]\!]$

$\lim_{x\to k^+} [\![x + 3]\!] = k + 3$, where k is an integer.

$\lim_{x\to k^-} [\![x + 3]\!] = k + 2$, where k is an integer.

f has a nonremovable discontinuity at each integer k, so f is continuous on $(k, k + 1)$ for all integers k.

57. $g(x) = 2e^{[\![x]\!]/4}$ is continuous on all intervals $(k, k + 1)$, where k is an integer. g has nonremovable discontinuities at each k.

58. $h(x) = -5 \ln|2 - x|$

Because $|2 - x| > 0$ except for $x = 2$, h is continuous on $(-\infty, 2) \cup (2, \infty)$.

59. $f(x) = \dfrac{3x^2 - x - 2}{x - 1} = \dfrac{(3x + 2)(x - 1)}{x - 1}$

$\lim_{x\to 1} f(x) = \lim_{x\to 1} (3x + 2) = 5$

f has a removable discontinuity at $x = 1$, so f is continuous on $(-\infty, 1) \cup (1, \infty)$.

60. $f(x) = \begin{cases} 5 - x, & x \le 2 \\ 2x - 3, & x > 2 \end{cases}$

$\lim_{x\to 2^-} (5 - x) = 3$

$\lim_{x\to 2^+} (2x - 3) = 1$

f has a nonremovable discontinuity at $x = 2$, so f is continuous on $(-\infty, 2) \cup (2, \infty)$.

61. Because $f(x)$ is continuous on the interval $[0, 1]$ and $f(0) = -2$ and $f(1) = 1$, by the Intermediate Value Theorem there exists a real number c in $[0, 1]$ such that $f(c) = 0$.

62. Because $f(x)$ is continuous on the interval $[3, 4]$ and $f(3) \approx -0.108$ and $f(4) \approx 0.159$, by the Intermediate Value Theorem there exists a real number c in $[3, 4]$ such that $f(c) = 0$.

63. $f(x) = \dfrac{x^3}{x^2 - 9} = \dfrac{x^3}{(x + 3)(x - 3)}$

$\lim_{x\to -3^-} \dfrac{x^3}{x^2 - 9} = -\infty$ and $\lim_{x\to -3^+} \dfrac{x^3}{x^2 - 9} = \infty$

Therefore, $x = -3$ is a vertical asymptote.

$\lim_{x\to -3^-} \dfrac{x^3}{x^2 - 9} = -\infty$ and $\lim_{x\to 3^+} \dfrac{x^3}{x^2 - 9} = \infty$

Therefore, $x = 3$ is a vertical asymptote.

64. $h(x) = \dfrac{12x}{144 - x^2} = \dfrac{12x}{(12 - x)(12 + x)}$

$\lim_{x\to -12^-} h(x) = \infty$ and $\lim_{x\to -12^+} h(x) = -\infty$

Therefore, $x = -12$ is a vertical asymptote.

$\lim_{x\to 12^-} h(x) = \infty$ and $\lim_{x\to 12^+} h(x) = -\infty$

Therefore, $x = 12$ is a vertical asymptote.

65. $g(x) = \dfrac{2x + 1}{4x^2 - 1} = \dfrac{2x + 1}{(2x + 1)(2x - 1)} = \dfrac{1}{2x - 1}$, $x \ne -\dfrac{1}{2}$

$\lim_{x\to \frac{1}{2}^-} g(x) = -\infty$ and $\lim_{x\to \frac{1}{2}^+} g(x) = \infty$

Therefore, $x = \dfrac{1}{2}$ is a vertical asymptote.

(**Note:** g has a removable discontinuity at $x = -\frac{1}{2}$.)

66. $f(x) = \csc \dfrac{\pi x}{3} = \dfrac{1}{\sin \dfrac{\pi x}{3}}$

$\sin \dfrac{\pi x}{3} = 0$ for $x = 3n$, where n is an integer.

$\lim_{x\to 3n^-} f(x) = \infty$ or $-\infty$ and $\lim_{x\to 3n^+} f(x) = \infty$ or $-\infty$

Therefore, $x = 3n$ is a vertical asymptote, where n is an integer.

67. $g(x) = \ln(25 - x^2) = \ln[(5 + x)(5 - x)]$

$\lim_{x\to 5} \ln(25 - x^2) = -\infty$

$\lim_{x\to -5} \ln(25 - x^2) = -\infty$

Therefore, the graph has vertical asymptotes at $x = \pm 5$.

68. $f(x) = 7e^{-3/x}$

$\lim_{x\to 0^-} 7e^{-3/x} = \infty$

Therefore, $x = 0$ is a vertical asymptote.

69. $\lim_{x\to 1^-} \dfrac{x^2 + 2x + 1}{x - 1} = -\infty$

70. $\lim_{x\to (1/2)^+} \dfrac{x}{2x - 1} = \infty$

71. $\lim_{x\to-2^+} \frac{x+2}{x^3+8} = \lim_{x\to-2^+} \frac{x+2}{(x+2)(x^2-2x+4)}$

$= \lim_{x\to-2^+} \frac{1}{x^2-2x+4}$

$= \frac{1}{(-2)^2-2(-2)+4}$

$= \frac{1}{12}$

72. $\lim_{x\to-1^-} \frac{x^2-1}{x^4-1} = \lim_{x\to-1^-} \frac{x^2-1}{(x^2-1)(x^2+1)}$

$= \lim_{x\to-1^-} \frac{1}{x^2+1}$

$= \frac{1}{(-1)^2+1}$

$= \frac{1}{2}$

73. $\lim_{x\to0^+} \left(x - \frac{1}{x^3}\right) = -\infty$

74. $\lim_{x\to2^-} \frac{1}{\sqrt[3]{x^2-4}} = -\infty$

75. $\lim_{x\to0^+} \frac{\sin 4x}{5x} = \lim_{x\to0^+} \left[\frac{4}{5}\left(\frac{\sin 4x}{4x}\right)\right] = \frac{4}{5}$

76. $\lim_{x\to0^+} \frac{\sec x}{x} = \infty$

77. $\lim_{x\to0^+} \ln(\sin x) = -\infty$

78. $\lim_{x\to0^-} 16e^{-2/x} = 16(\infty) = \infty$

79. $C = \frac{80{,}000p}{100-p}, 0 \le p < 100$

(a) $C(15) \approx \$14{,}117.65$

$C(50) = \$80{,}000$

$C(90) = \$720{,}000$

(b) $\lim_{p\to100^-} \frac{80{,}000p}{100-p} = \infty$

No matter how much the company spends, the company will never be able to remove 100% of the pollutants.

80. $f(x) = \frac{\tan 2x}{x}$

(a)

x	–0.1	–0.01	–0.001	0.001	0.01	0.1
$f(x)$	2.0271	2.0003	2.0000	2.0000	2.0003	2.0271

$\lim_{x\to0} \frac{\tan 2x}{x} = 2$

(b) Yes, define $f(x) = \begin{cases} \frac{\tan 2x}{x}, & x \ne 0 \\ 2, & x = 0 \end{cases}$.

So, $f(x)$ is continuous at $x = 0$.

81. $\lim_{x\to\infty} \left(8 + \frac{1}{x}\right) = 8 + 0 = 8$

82. $\lim_{x\to-\infty} \frac{1-4x}{x+1} = \lim_{x\to-\infty} \frac{(1/x)-4}{1+4x} = -4$

83. $\lim_{x\to\infty} \frac{3x^2}{5x^2+2} = \lim_{x\to\infty} \frac{3}{5+\frac{2}{x^2}} = \frac{3}{5+0} = \frac{3}{5}$

84. $\lim_{x\to\infty} \frac{4x^3}{x^4+3} = \lim_{x\to\infty} \frac{4/x}{1+(3/x^4)} = 0$

85. $\lim_{x\to-\infty} \frac{3x^2}{x+5} = -\infty$

86. $\lim_{x\to-\infty} \frac{\sqrt{x^2+x}}{-2x} = 1/2$

87. $\lim_{x\to-\infty} \frac{6x}{x+\cos x} = 6$

88. $\lim_{x\to-\infty} \frac{x}{2\sin x}$ does not exist.

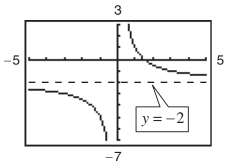
89. $f(x) = \dfrac{3}{x} - 2$

Discontinuity: $x = 0$

$$\lim_{x\to\infty}\left(\frac{3}{x} - 2\right) = -2$$

Vertical asymptote: $x = 0$

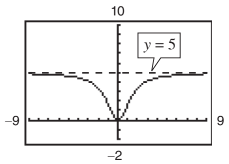
Horizontal asymptote: $y = -2$



90. $g(x) = \dfrac{5x^2}{x^2 + 2}$

$$\lim_{x\to\infty}\frac{5x^2}{x^2 + 2} = \lim_{x\to\infty}\frac{5}{1 + \left(2/x^2\right)} = 5$$

Horizontal asymptote: $y = 5$



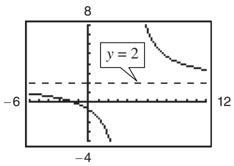
91. $h(x) = \dfrac{2x + 3}{x - 4}$

Discontinuity: $x = 4$

$$\lim_{x\to\infty}\frac{2x + 3}{x - 4} = \lim_{x\to\infty}\frac{2 + (3/x)}{1 - (4/x)} = 2$$

Vertical asymptote: $x = 4$

Horizontal asymptote: $y = 2$

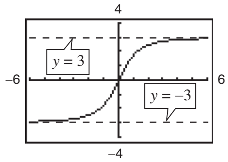


92. $f(x) = \dfrac{3x}{\sqrt{x^2 + 2}}$

$$\lim_{x\to\infty}\frac{3x}{\sqrt{x^2 + 2}} = \lim_{x\to\infty}\frac{3x/x}{\sqrt{x^2 + 2}\big/\sqrt{x^2}} = \lim_{x\to\infty}\frac{3}{\sqrt{1 + \left(2/x^2\right)}} = 3$$

$$\lim_{x\to-\infty}\frac{3x}{\sqrt{x^2 + 2}} = \lim_{x\to-\infty}\frac{3x/x}{\sqrt{x^2 + 2}\big/\left(-\sqrt{x^2}\right)} = \lim_{x\to-\infty}\frac{3}{-\sqrt{1 + \left(2/x^2\right)}} = -3$$

Horizontal asymptotes: $y = \pm 3$

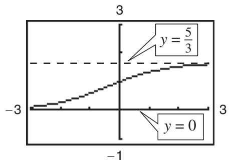


93. $f(x) = \dfrac{5}{3 + 2e^{-x}}$

$$\lim_{x\to\infty}\frac{5}{3 + 2e^{-x}} = \frac{5}{3}$$

$$\lim_{x\to-\infty}\frac{5}{3 + 2e^{-x}} = 0$$

Horizontal asymptotes: $y = 0, y = \dfrac{5}{3}$



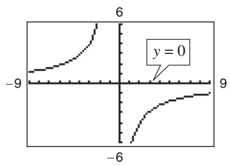
94. $h(x) = 10\ln\left(\dfrac{x}{x + 1}\right)$

Discontinuities: $x = 0, x = -1$

$$\lim_{x\to\infty}10\ln\left(\frac{x}{x + 1}\right) = \lim_{x\to-\infty}10\ln\left(\frac{x}{x + 1}\right) = 0$$

Vertical asymptotes: $x = 0, x = -1$

Horizontal asymptote: $y = 0$



AP® Exam Practice Questions for Chapter 1

1. $\lim_{x\to\pi} h(x) = \lim_{x\to\pi} 2 = 2$

The answer is B.

2. $\lim_{x\to-4^-} g(x) = \lim_{x\to-4^-} \dfrac{|x+4|}{x+4} = -1$

$\lim_{x\to-4^+} g(x) = \lim_{x\to-4^+} \dfrac{|x+4|}{x+4} = 1$

Because $\lim_{x\to-4^-} g(x) \neq \lim_{x\to-4^+}$, the limit is nonexistent.

The answer is D.

3. $\lim_{x\to\pi} \dfrac{\sin x}{x} = \dfrac{\sin \pi}{\pi} = \dfrac{0}{\pi} = 0$

The answer is A.

4. $\lim_{x\to-2} \dfrac{3x^2 + 5x + 7}{x - 4}$

$= \dfrac{3(-2)^2 + 5(-2) + 7}{(-2) - 4}$

$= \dfrac{9}{-6}$

$= -\dfrac{3}{2}$

The answer is B.

5. $\lim_{x\to5}\left[5f(x) - g(x)\right] = \lim_{x\to5} 5f(x) - \lim_{x\to5} g(x)$

$= 5\lim_{x\to5} f(x) - \lim_{x\to5} g(x)$

$= 5(10) - (1) = 49$

The answer is D.

6. $\lim_{x\to\infty} \dfrac{5x + 16x^2}{4x^2 - 3} = \lim_{x\to\infty} \dfrac{\frac{5}{x} + 16}{4 - \frac{3}{x^2}}$

$= \dfrac{0 + 16}{4 - 0}$

$= 4$

The answer is C.

7. $\lim_{x\to\infty} \dfrac{3}{1 - 4^x} = 0$

$\lim_{x\to-\infty} \dfrac{3}{1 - 4^x} = \dfrac{3}{1 - 0} = 3$

The answer is C.

8. Evaluate each statement.

I. Because $\lim_{x\to2^-} g(x) = 1$ and $\lim_{x\to2^+} g(x) = 1$, $\lim_{x\to2} g(x) = 1$.

The statement is true.

II. $\lim_{x\to2} g(x) = 1 \neq g(2) = 3$

The statement is false.

III. g is continuous at $x = 3$.

The statement is true.

Because I and III are true, the answer is B.

9. $\lim_{x\to\infty} \dfrac{\sqrt{9x^4 - 5}}{5x - 3x^2} = \lim_{x\to\infty} \dfrac{\dfrac{\sqrt{9x^4 - 5}}{\sqrt{x^4}}}{\dfrac{5x - 3x^2}{x^2}}$

$= \lim_{x\to\infty} \dfrac{\sqrt{9 - \frac{5}{x^4}}}{\frac{5}{x} - 3}$

$= \dfrac{\sqrt{9 - 0}}{0 - 3}$

$= -1$

The answer is C.

10. Because $\lim_{x\to1^-} \dfrac{x - 1}{\sqrt{x} - 1} = 2$ and $\lim_{x\to1^+} \dfrac{x - 1}{\sqrt{x} - 1} = 2$, $\lim_{x\to1} \dfrac{x - 1}{\sqrt{x} - 1} = 2$.

The answer is C.

11. (a) $s(t)$ is continuous on $[1, 2]$.

$s(1) = 393.1$ and $s(2) = 378.4$

Because 382 is between $s(1)$ and $s(2)$, by the Intermediate Value Theorem there exists at least one value of t in $[1, 2]$ such that $s(t) = 382$.

(b) $s(t) = -4.9t^2 + 398$

$0 = -4.9t^2 + 398$

$t \approx \pm 9.012$

So, the object hits the ground after approximately 9.012 seconds.

(c)
$$\begin{aligned}\lim_{t\to 3}\frac{s(t) - s(3)}{t - 3} &= \lim_{t\to 3}\frac{\left(-4.9t^2 + 398\right) - \left[-4.9(3)^2 + 398\right]}{t - 3}\\ &= \lim_{t\to 3}\frac{-4.9t^2 + 4.9(9)}{t - 3}\\ &= \lim_{t\to 3}\frac{-4.9\left(t^2 - 9\right)}{t - 3}\\ &= \lim_{t\to 3}\frac{-4.9(t - 3)(t + 3)}{t - 3}\\ &= \lim_{t\to 3} -4.9(t + 3)\\ &= -4.9(3 + 3)\\ &= -29.4 \text{ m/sec}\end{aligned}$$

12. (a) $\displaystyle\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{10}{1 + \frac{1}{4}e^{-x}} = \frac{10}{1 + \frac{1}{4}} = \frac{10}{\frac{5}{4}} = 8$

(b) $\displaystyle\lim_{x\to 0}\left[f(x) + 4\right] = \lim_{x\to 0} f(x) + \lim_{x\to 0} 4 = 8 + 4 = 12$

(c) $\displaystyle\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{10}{1 + \frac{1}{4}e^{-x}} = \lim_{x\to\infty}\frac{10}{1 + \frac{1}{4e^x}} = \frac{10}{1 + \frac{1}{4e^\infty}} = \frac{10}{1 + 0} = 10$

$\displaystyle\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty}\frac{10}{1 + \frac{1}{4}e^{-x}} = \frac{10}{1 + \frac{1}{4}e^\infty} = \frac{10}{\infty} = 0$

So, the horizontal asymptotes are $y = 0$ and $y = 10$.

13. (a) $\displaystyle f(x) = \frac{x^2 + 5x + 6}{2x^2 + 7x + 3} = \frac{(x + 2)(x + 3)}{(2x + 1)(x + 3)} = \frac{x + 2}{2x + 1}, x \neq -3$

$f(x)$ has discontinuities at $x = -\frac{1}{2}$ and $x = -3$.

(b)
$$\begin{aligned}\lim_{x\to -3} f(x) &= \lim_{x\to -3}\frac{x^2 + 5x + 6}{2x^2 + 7x + 3} = \lim_{x\to -3}\frac{(x + 2)\cancel{(x + 3)}}{(2x + 1)\cancel{(x + 3)}}\\ &= \lim_{x\to -3}\frac{x + 2}{2x + 1} = \frac{-3 + 2}{2(-3) + 1} = \frac{1}{5}\end{aligned}$$

(c) $f(x)$ has a vertical asymptote at $x = -\frac{1}{2}$.

(d) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{x^2 + 5x + 6}{2x^2 + 7x + 3} = \lim_{x\to\infty} \dfrac{1 + \frac{5}{x} + \frac{6}{x^2}}{2 + \frac{7}{x} + \frac{3}{x^2}} = \dfrac{1 + 0 + 0}{2 + 0 + 0} = \dfrac{1}{2}$

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \dfrac{x^2 + 5x + 6}{2x^2 + 7x + 3} = \lim_{x\to-\infty} \dfrac{1 + \frac{5}{x} + \frac{6}{x^2}}{2 + \frac{7}{x} + \frac{3}{x^2}} = \dfrac{1 + 0 + 0}{2 + 0 + 0} = \dfrac{1}{2}$

$f(x)$ has a horizontal asymptote at $y = \dfrac{1}{2}$.

14. (a) $\lim_{x\to-1} f(x) = \lim_{x\to-1} e^{2x} = e^{2(-1)} = \dfrac{1}{e^2}$

(b) $f(0)$ is defined as $f(0) = e^{2(0)} = 1$. $\lim_{x\to0^-} f(x) = 1$ and $\lim_{x\to0^+} f(x) = 1$, so $\lim_{x\to0} f(x) = 1$.

Also, $\lim_{x\to0} f(x) = f(0) = 1$.

So, f is continuous at $x = 0$.

(c) $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} e^{2x} = e^{2(-\infty)} = \dfrac{1}{e^\infty} = \dfrac{1}{\infty} = 0$

15. (a) $\lim_{x\to1} \left[f(x) + 4\right] = \lim_{x\to1} f(x) + \lim_{x\to1} 4 = 2 + 4 = 6$

(b) $\lim_{x\to3^-} \dfrac{5}{g(x)} = \dfrac{5}{1} = 5$

(c) $\lim_{x\to2} \left[f(x)g(x)\right] = 2 \cdot 0 = 0$

(d) $\lim_{x\to3} \dfrac{f(x)}{g(x) - 1} = \lim_{x\to3} \dfrac{(-2x + 6)}{(x - 2) - 1} = \lim_{x\to3} \dfrac{-2x + 6}{x - 3} = \lim_{x\to3} \dfrac{-2\cancel{(x - 3)}}{\cancel{(x - 3)}} = -2$

16. (a) Because $T(x)$ is continuous on $[0, 10)$, $\lim_{x\to4} T(x) = T(4) = 172$.

(b) $\dfrac{T(8) - T(3)}{8 - 3} = \dfrac{164 - 174}{8 - 3} = \dfrac{-10}{5} = -2$

The average rate of change is -2°F per minute.

(c) $T(x)$ is continuous and when $x = 6$, $T(x) > 166.5°$ and when $x = 8$, $T(x) < 166.5°$.

So, the shortest interval is $(6, 8)$.

(d) Because $T(x)$ is continuous, the average rate of change for $6 \le x \le 9$ is

$\dfrac{T(9) - T(6)}{9 - 6} = \dfrac{162 - 168}{9 - 6} = \dfrac{-6}{3} = -2.$

So, the tangent line at $x = 8$ has a slope of about -2.

17. (a) $f(x) = ax^2 + x - b$ $\quad$ $f(x) = ax + b$

$f(2) = a(2)^2 + (2) - b$ $\quad$ $f(2) = a(2) + b$

$= 4a + 2 - b$ $\quad$ $= 2a + b$

f is continuous at $x = 2$ when $4a + 2 - b = 2a + b$.

$f(x) = ax + b$ $\quad$ $f(x) = 2ax - 7$

$f(5) = a(5) + b$ $\quad$ $f(5) = 2a(5) - 7$

$= 5a + b$ $\quad$ $= 10a - 7$

f is continuous at $x = 5$ when $5a + b = 10a - 7$.

$4a + 2 - b = 2a + b \Rightarrow 2a - 2b = -2$

$5a + b = 10a - 7 \Rightarrow -5a + b = -7$

Multiply both sides of the second equation by 2.

$$\begin{array}{rcl} 2a - 2b &=& -2 \\ -10a + 2b &=& -14 \\ \hline -8a &=& -16 \\ a &=& 2 \end{array}$$

When $a = 2$, $b = 5(2) - 7 = 10 - 7 = 3$.

So, f is continuous when $a = 2$ and $b = 3$.

(b) $\lim_{x\to 3} f(x) = \lim_{x\to 3}(2x + 3) = 2(3) + 3 = 9$

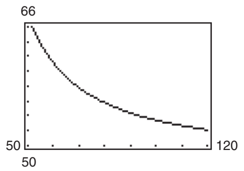
(c) $\lim_{x\to 1} g(x) = \lim_{x\to 1}\dfrac{f(x)}{x - 1} = \lim_{x\to 1}\dfrac{2x^2 + x - 3}{x - 1} = \lim_{x\to 1}\dfrac{(2x + 3)(x - 1)}{x - 1} = \lim_{x\to 1}(2x + 3) = 2(1) + 3 = 5$

Performance Task for Chapter 1

1. Answers will vary.

2. Answers will vary. Sample answer: The women's 100-meter freestyle record in 2025 should be about 51 seconds because as time goes on, the record time decreases at a slower rate.

3. (a)



When $x = 125$, $y = \dfrac{32.4 - 1.66(125)}{1 - 0.035(125)} \approx 51.88$

In 2025, the record time will be about 51.88 seconds.

(b) $\lim_{x\to\infty}\dfrac{32.4 - 1.66x}{1 - 0.035x} = \lim_{x\to\infty}\dfrac{\dfrac{32.4}{x} - 1.66}{\dfrac{1}{x} - 0.035}$

$= \dfrac{-1.66}{-0.035}$

≈ 47.43

As $x \to \infty$, the record time approaches 47.43 seconds.

4. Answers will vary.

5. Answers will vary. Sample answer: Logic tells us there must be a lower limit and the limit must not be zero. These results only show that there is a model for the years given with a lower limit.