CHAPTER ONE

Solutions for Section 1.1

Exercises

1. Since t represents the number of years since 1970, we see that $f(35)$ represents the population of the city in 2005. In 2005, the city's population was 12 million.

2. Since $T = f(P)$, we see that $f(200)$ is the value of T when $P = 200$; that is, the thickness of pelican eggs when the concentration of PCBs is 200 ppm.

3. If there are no workers, there is no productivity, so the graph goes through the origin. At first, as the number of workers increases, productivity also increases. As a result, the curve goes up initially. At a certain point the curve reaches its highest level, after which it goes downward; in other words, as the number of workers increases beyond that point, productivity decreases. This might, for example, be due either to the inefficiency inherent in large organizations or simply to workers getting in each other's way as too many are crammed on the same line. Many other reasons are possible.

4. The slope is $(1 - 0)/(1 - 0) = 1$. So the equation of the line is $y = x$.

5. The slope is $(3 - 2)/(2 - 0) = 1/2$. So the equation of the line is $y = (1/2)x + 2$.

6. Using the points $(-2, 1)$ and $(2, 3)$, we have

$$\text{Slope } = \frac{3-1}{2-(-2)} = \frac{2}{4} = \frac{1}{2}.$$

Now we know that $y = (1/2)x + b$. Using the point $(-2, 1)$, we have $1 = -2/2 + b$, which yields $b = 2$. Thus, the equation of the line is $y = (1/2)x + 2$.

7. Slope = $\dfrac{6-0}{2-(-1)} = 2$ so the equation is $y - 6 = 2(x - 2)$ or $y = 2x + 2$.

8. Rewriting the equation as $y = -\dfrac{5}{2}x + 4$ shows that the slope is $-\dfrac{5}{2}$ and the vertical intercept is 4.

9. Rewriting the equation as

$$y = -\frac{12}{7}x + \frac{2}{7}$$

shows that the line has slope $-12/7$ and vertical intercept $2/7$.

10. Rewriting the equation of the line as

$$\begin{aligned} -y &= \frac{-2}{4}x - 2 \\ y &= \frac{1}{2}x + 2, \end{aligned}$$

we see the line has slope $1/2$ and vertical intercept 2.

11. Rewriting the equation of the line as

$$\begin{aligned} y &= \frac{12}{6}x - \frac{4}{6} \\ y &= 2x - \frac{2}{3}, \end{aligned}$$

we see that the line has slope 2 and vertical intercept $-2/3$.

12. (a) is (V), because slope is positive, vertical intercept is negative
(b) is (IV), because slope is negative, vertical intercept is positive
(c) is (I), because slope is 0, vertical intercept is positive
(d) is (VI), because slope and vertical intercept are both negative
(e) is (II), because slope and vertical intercept are both positive
(f) is (III), because slope is positive, vertical intercept is 0

13. (a) is (V), because slope is negative, vertical intercept is 0
(b) is (VI), because slope and vertical intercept are both positive
(c) is (I), because slope is negative, vertical intercept is positive
(d) is (IV), because slope is positive, vertical intercept is negative
(e) is (III), because slope and vertical intercept are both negative
(f) is (II), because slope is positive, vertical intercept is 0

14. The intercepts appear to be $(0, 3)$ and $(7.5, 0)$, giving

$$\text{Slope } = \frac{-3}{7.5} = -\frac{6}{15} = -\frac{2}{5}.$$

The y-intercept is at $(0, 3)$, so a possible equation for the line is

$$y = -\frac{2}{5}x + 3.$$

(Answers may vary.)

15. $y - c = m(x - a)$

16. Given that the function is linear, choose any two points, for example (5.2, 27.8) and (5.3, 29.2). Then

$$\text{Slope} = \frac{29.2 - 27.8}{5.3 - 5.2} = \frac{1.4}{0.1} = 14.$$

Using the point-slope formula, with the point $(5.2, 27.8)$, we get the equation

$$y - 27.8 = 14(x - 5.2)$$

which is equivalent to

$$y = 14x - 45.$$

17. $y = 5x - 3$. Since the slope of this line is 5, we want a line with slope $-\frac{1}{5}$ passing through the point $(2, 1)$. The equation is $(y - 1) = -\frac{1}{5}(x - 2)$, or $y = -\frac{1}{5}x + \frac{7}{5}$.

18. The line $y + 4x = 7$ has slope -4. Therefore the parallel line has slope -4 and equation $y - 5 = -4(x - 1)$ or $y = -4x + 9$. The perpendicular line has slope $\frac{-1}{(-4)} = \frac{1}{4}$ and equation $y - 5 = \frac{1}{4}(x - 1)$ or $y = 0.25x + 4.75$.

19. The line parallel to $y = mx + c$ also has slope m, so its equation is

$$y = m(x - a) + b.$$

The line perpendicular to $y = mx + c$ has slope $-1/m$, so its equation will be

$$y = -\frac{1}{m}(x - a) + b.$$

20. Since the function goes from $x = 0$ to $x = 4$ and between $y = 0$ and $y = 2$, the domain is $0 \leq x \leq 4$ and the range is $0 \leq y \leq 2$.

21. Since x goes from 1 to 5 and y goes from 1 to 6, the domain is $1 \leq x \leq 5$ and the range is $1 \leq y \leq 6$.

22. Since the function goes from $x = -2$ to $x = 2$ and from $y = -2$ to $y = 2$, the domain is $-2 \leq x \leq 2$ and the range is $-2 \leq y \leq 2$.

23. Since the function goes from $x = 0$ to $x = 5$ and between $y = 0$ and $y = 4$, the domain is $0 \leq x \leq 5$ and the range is $0 \leq y \leq 4$.

24. The domain is all numbers. The range is all numbers ≥ 2, since $x^2 \geq 0$ for all x.

25. The domain is all x-values, as the denominator is never zero. The range is $0 < y \leq \dfrac{1}{2}$.

26. The value of $f(t)$ is real provided $t^2 - 16 \geq 0$ or $t^2 \geq 16$. This occurs when either $t \geq 4$, or $t \leq -4$. Solving $f(t) = 3$, we have

$$\begin{aligned} \sqrt{t^2 - 16} &= 3 \\ t^2 - 16 &= 9 \\ t^2 &= 25 \end{aligned}$$

so

$$t = \pm 5.$$

27. We have $V = kr^3$. You may know that $V = \frac{4}{3}\pi r^3$.

28. If distance is d, then $v = \frac{d}{t}$.

29. For some constant k, we have $S = kh^2$.

30. We know that E is proportional to v^3, so $E = kv^3$, for some constant k.

31. We know that N is proportional to $1/l^2$, so

$$N = \frac{k}{l^2}, \quad \text{for some constant } k.$$

Problems

32. The year 1983 was 25 years before 2008 so 1983 corresponds to $t = 25$. Thus, an expression that represents the statement is:

$$f(25) = 7.019$$

33. The year 2008 was 0 years before 2008 so 2008 corresponds to $t = 0$. Thus, an expression that represents the statement is:

$$f(0) \text{ meters.}$$

34. The year 1965 was $2008 - 1865 = 143$ years before 2008 so 1965 corresponds to $t = 143$. Similarly, we see that the year 1911 corresponds to $t = 97$. Thus, an expression that represents the statement is:

$$f(143) = f(97)$$

35. Since $t = 1$ means one year before 2008, then $t = 1$ corresponds to the year 2007. Similarly, $t = 0$ corresponds to the year 2008. Thus, $f(1)$ and $f(0)$ are the average annual sea level values, in meters, in 2007 and 2008, respectively. Because 1 millimeter is the same as 0.001 meters, an expression that represents the statement is:

$$f(0) = f(1) + 0.001.$$

Note that there are other possible equivalent expressions, such as: $f(1) - f(0) = 0.001$.

36. (a) Each date, t, has a unique daily snowfall, S, associated with it. So snowfall is a function of date.
(b) On December 12, the snowfall was approximately 5 inches.
(c) On December 11, the snowfall was above 10 inches.
(d) Looking at the graph we see that the largest increase in the snowfall was between December 10 to December 11.

37. (a) When the car is 5 years old, it is worth \$6000.
(b) Since the value of the car decreases as the car gets older, this is a decreasing function. A possible graph is in Figure 1.1:

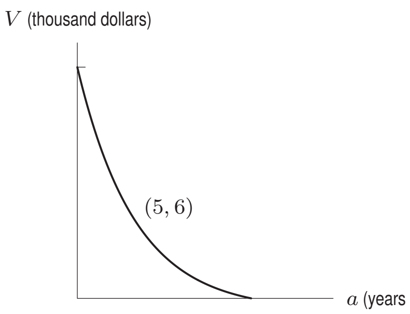


Figure 1.1

(c) The vertical intercept is the value of V when $a = 0$, or the value of the car when it is new. The horizontal intercept is the value of a when $V = 0$, or the age of the car when it is worth nothing.

38. (a) The story in (a) matches Graph (IV), in which the person forgot her books and had to return home.
(b) The story in (b) matches Graph (II), the flat tire story. Note the long period of time during which the distance from home did not change (the horizontal part).
(c) The story in (c) matches Graph (III), in which the person started calmly but sped up later.

The first graph (I) does not match any of the given stories. In this picture, the person keeps going away from home, but his speed decreases as time passes. So a story for this might be: *I started walking to school at a good pace, but since I stayed up all night studying calculus, I got more and more tired the farther I walked.*

39. (a) $f(30) = 10$ means that the value of f at $t = 30$ was 10. In other words, the temperature at time $t = 30$ minutes was 10°C. So, 30 minutes after the object was placed outside, it had cooled to 10 °C.
(b) The intercept a measures the value of $f(t)$ when $t = 0$. In other words, when the object was initially put outside, it had a temperature of a°C. The intercept b measures the value of t when $f(t) = 0$. In other words, at time b the object's temperature is 0 °C.

40. (a) The height of the rock decreases as time passes, so the graph falls as you move from left to right. One possibility is shown in Figure 1.2.

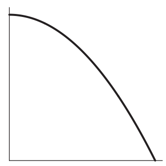


Figure 1.2

(b) The statement $f(7) = 12$ tells us that 7 seconds after the rock is dropped, it is 12 meters above the ground.
(c) The vertical intercept is the value of s when $t = 0$; that is, the height from which the rock is dropped. The horizontal intercept is the value of t when $s = 0$; that is, the time it takes for the rock to hit the ground.

41. (a) We find the slope m and intercept b in the linear equation $C = b + mw$. To find the slope m, we use

$$m = \frac{\Delta C}{\Delta w} = \frac{12.32 - 8}{68 - 32} = 0.12 \text{ dollars per gallon.}$$

We substitute to find b:

$$\begin{aligned} C &= b + mw \\ 8 &= b + (0.12)(32) \\ b &= 4.16 \text{ dollars.} \end{aligned}$$

The linear formula is $C = 4.16 + 0.12w$.
(b) The slope is 0.12 dollars per gallon. Each additional gallon of waste collected costs 12 cents.
(c) The intercept is \$4.16. The flat monthly fee to subscribe to the waste collection service is \$4.16. This is the amount charged even if there is no waste.

42. We are looking for a linear function $y = f(x)$ that, given a time x in years, gives a value y in dollars for the value of the refrigerator. We know that when $x = 0$, that is, when the refrigerator is new, $y = 950$, and when $x = 7$, the refrigerator is worthless, so $y = 0$. Thus $(0, 950)$ and $(7, 0)$ are on the line that we are looking for. The slope is then given by

$$m = \frac{950}{-7}$$

It is negative, indicating that the value decreases as time passes. Having found the slope, we can take the point $(7, 0)$ and use the point-slope formula:

$$y - y_1 = m(x - x_1).$$

So,

$$\begin{aligned} y - 0 &= -\frac{950}{7}(x - 7) \\ y &= -\frac{950}{7}x + 950. \end{aligned}$$

43. (a) The first company's price for a day's rental with m miles on it is $C_1(m) = 40 + 0.15m$. Its competitor's price for a day's rental with m miles on it is $C_2(m) = 50 + 0.10m$.

(b) See Figure 1.3.

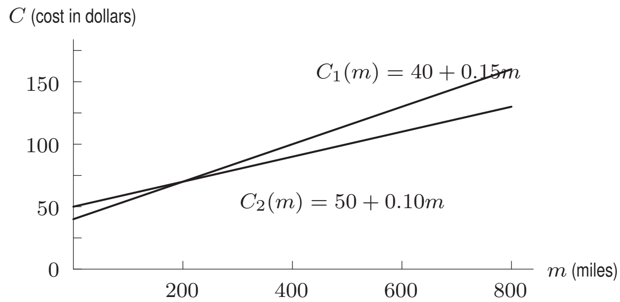


Figure 1.3

(c) To find which company is cheaper, we need to determine where the two lines intersect. We let $C_1 = C_2$, and thus

$$\begin{aligned} 40 + 0.15m &= 50 + 0.10m \\ 0.05m &= 10 \\ m &= 200. \end{aligned}$$

If you are going more than 200 miles a day, the competitor is cheaper. If you are going less than 200 miles a day, the first company is cheaper.

44. (a) Charge per cubic foot = $\dfrac{\Delta \$}{\Delta \text{ cu. ft.}} = \dfrac{55 - 40}{1600 - 1000} = \$0.025/\text{cu. ft.}$

Alternatively, if we let c = cost, w = cubic feet of water, b = fixed charge, and m = cost/cubic feet, we obtain $c = b + mw$. Substituting the information given in the problem, we have

$$\begin{aligned} 40 &= b + 1000m \\ 55 &= b + 1600m. \end{aligned}$$

Subtracting the first equation from the second yields $15 = 600m$, so $m = 0.025$.

(b) The equation is $c = b + 0.025w$, so $40 = b + 0.025(1000)$, which yields $b = 15$. Thus the equation is $c = 15 + 0.025w$.

(c) We need to solve the equation $100 = 15 + 0.025w$, which yields $w = 3400$. It costs \$100 to use 3400 cubic feet of water.

45. See Figure 1.4.

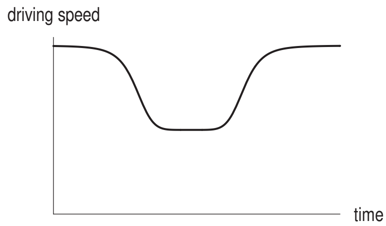


Figure 1.4

46. See Figure 1.5.

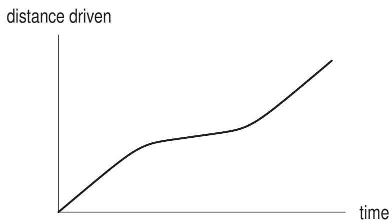


Figure 1.5

47. See Figure 1.6.

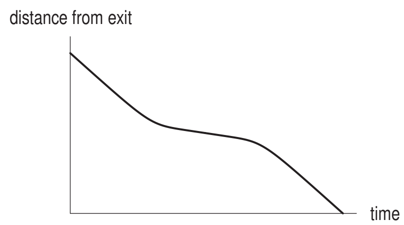


Figure 1.6

48. See Figure 1.7.

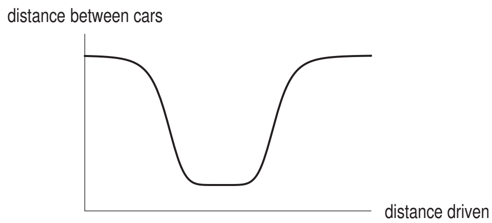


Figure 1.7

49. (a) (i) $f(1985) = 13$
(ii) $f(1990) = 99$
(b) The average yearly increase is the rate of change.

$$\text{Yearly increase} = \frac{f(1990) - f(1985)}{1990 - 1985} = \frac{99 - 13}{5} = 17.2 \text{ billionaires per year.}$$

(c) Since we assume the rate of increase remains constant, we use a linear function with slope 17.2 billionaires per year. The equation is

$$f(t) = b + 17.2t$$

where $f(1985) = 13$, so

$$13 = b + 17.2(1985)$$
$$b = -34{,}129.$$

Thus, $f(t) = 17.2t - 34{,}129$.

50. (a) The largest time interval was 2008–2009 since the percentage growth rate increased from -11.7 to 7.3 from 2008 to 2009. This means the US consumption of biofuels grew relatively more from 2008 to 2009 than from 2007 to 2008. (Note that the percentage growth rate was a decreasing function of time over 2005–2007.)
(b) The largest time interval was 2005–2007 since the percentage growth rates were positive for each of these three consecutive years. This means that the amount of biofuels consumed in the US steadily increased during the three year span from 2005 to 2007, then decreased in 2008.

51. (a) The largest time interval was 2005–2007 since the percentage growth rate decreased from -1.9 in 2005 to -45.4 in 2007. This means that from 2005 to 2007 the US consumption of hydroelectric power shrunk relatively more with each successive year.
(b) The largest time interval was 2004–2007 since the percentage growth rates were negative for each of these four consecutive years. This means that the amount of hydroelectric power consumed by the US industrial sector steadily decreased during the four year span from 2004 to 2007, then increased in 2008.

52. (a) The largest time interval was 2004–2006 since the percentage growth rate increased from -5.7 in 2004 to 9.7 in 2006. This means that from 2004 to 2006 the US price per watt of a solar panel grew relatively more with each successive year.
(b) The largest time interval was 2005–2006 since the percentage growth rates were positive for each of these two consecutive years. This means that the US price per watt of a solar panel steadily increased during the two year span from 2005 to 2006, then decreased in 2007.

53. (a) Since 2008 corresponds to $t = 0$, the average annual sea level in Aberdeen in 2008 was 7.094 meters.
(b) Looking at the table, we see that the average annual sea level was 7.019 fifty years before 2008, or in the year 1958. Similar reasoning shows that the average sea level was 6.957 meters 125 years before 2008, or in 1883.
(c) Because 125 years before 2008 the year was 1883, we see that the sea level value corresponding to the year 1883 is 6.957 (this is the sea level value corresponding to $t = 125$). Similar reasoning yields the table:

Year	1883	1908	1933	1958	1983	2008
S	6.957	6.938	6.965	6.992	7.019	7.094

54. (a) We find the slope m and intercept b in the linear equation $S = b + mt$. To find the slope m, we use

$$m = \frac{\Delta S}{\Delta t} = \frac{66 - 113}{50 - 0} = -0.94.$$

When $t = 0$, we have $S = 113$, so the intercept b is 113. The linear formula is

$$S = 113 - 0.94t.$$

(b) We use the formula $S = 113 - 0.94t$. When $S = 20$, we have $20 = 113 - 0.94t$ and so $t = 98.9$. If this linear model were correct, the average male sperm count would drop below the fertility level during the year 2038.

55. (a) This could be a linear function because w increases by 5 as h increases by 1.
(b) We find the slope m and the intercept b in the linear equation $w = b + mh$. We first find the slope m using the first two points in the table. Since we want w to be a function of h, we take

$$m = \frac{\Delta w}{\Delta h} = \frac{171 - 166}{69 - 68} = 5.$$

Substituting the first point and the slope $m = 5$ into the linear equation $w = b + mh$, we have $166 = b + (5)(68)$, so $b = -174$. The linear function is

$$w = 5h - 174.$$

The slope, $m = 5$, is in units of pounds per inch.
(c) We find the slope and intercept in the linear function $h = b + mw$ using $m = \Delta h/\Delta w$ to obtain the linear function

$$h = 0.2w + 34.8.$$

Alternatively, we could solve the linear equation found in part (b) for h. The slope, $m = 0.2$, has units inches per pound.

56. We will let

$$\begin{aligned} T &= \text{ amount of fuel for take-off,} \\ L &= \text{ amount of fuel for landing,} \\ P &= \text{ amount of fuel per mile in the air,} \\ m &= \text{ the length of the trip in miles.} \end{aligned}$$

Then Q, the total amount of fuel needed, is given by

$$Q(m) = T + L + Pm.$$

57. (a) The variable costs for x acres are \$$200x$, or $0.2x$ thousand dollars. The total cost, C (again in thousands of dollars), of planting x acres is:

$$C = f(x) = 10 + 0.2x.$$

This is a linear function. See Figure 1.8. Since $C = f(x)$ increases with x, f is an increasing function of x. Look at the values of C shown in the table; you will see that each time x increases by 1, C increases by 0.2. Because C increases at a constant rate as x increases, the graph of C against x is a line.

(b) See Figure 1.8 and Table 1.1.

Table 1.1
Cost of planting seed

x	C
0	10
2	10.4
3	10.6
4	10.8
5	11
6	11.2

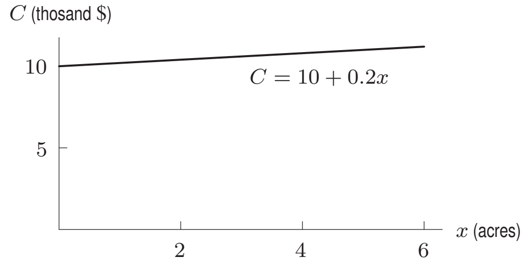


Figure 1.8

(c) The vertical intercept of 10 corresponds to the fixed costs. For $C = f(x) = 10 + 0.2x$, the intercept on the vertical axis is 10 because $C = f(0) = 10 + 0.2(0) = 10$. Since 10 is the value of C when $x = 0$, we recognize it as the initial outlay for equipment, or the fixed cost.

The slope 0.2 corresponds to the variable costs. The slope is telling us that for every additional acre planted, the costs go up by 0.2 thousand dollars. The rate at which the cost is increasing is 0.2 thousand dollars per acre. Thus the variable costs are represented by the slope of the line $f(x) = 10 + 0.2x$.

58. See Figure 1.9.

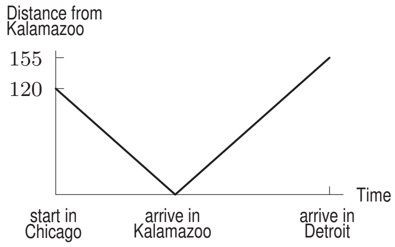


Figure 1.9

59. (a) The line given by $(0, 2)$ and $(1, 1)$ has slope $m = \frac{2-1}{-1} = -1$ and y-intercept 2, so its equation is

$$y = -x + 2.$$

The points of intersection of this line with the parabola $y = x^2$ are given by

$$\begin{aligned} x^2 &= -x + 2 \\ x^2 + x - 2 &= 0 \\ (x + 2)(x - 1) &= 0. \end{aligned}$$

The solution $x = 1$ corresponds to the point we are already given, so the other solution, $x = -2$, gives the x-coordinate of C. When we substitute back into either equation to get y, we get the coordinates for C, $(-2, 4)$.

(b) The line given by $(0, b)$ and $(1, 1)$ has slope $m = \frac{b-1}{-1} = 1 - b$, and y-intercept at $(0, b)$, so we can write the equation for the line as we did in part (a):

$$y = (1 - b)x + b.$$

We then solve for the points of intersection with $y = x^2$ the same way:

$$\begin{aligned} x^2 &= (1 - b)x + b \\ x^2 - (1 - b)x - b &= 0 \\ x^2 + (b - 1)x - b &= 0 \\ (x + b)(x - 1) &= 0 \end{aligned}$$

Again, we have the solution at the given point $(1, 1)$, and a new solution at $x = -b$, corresponding to the other point of intersection C. Substituting back into either equation, we can find the y-coordinate for C is b^2, and thus C is given by $(-b, b^2)$. This result agrees with the particular case of part (a) where $b = 2$.

60. Looking at the given data, it seems that Galileo's hypothesis was incorrect. The first table suggests that velocity is not a linear function of distance, since the increases in velocity for each foot of distance are themselves getting smaller. Moreover, the second table suggests that velocity is instead proportional to *time*, since for each second of time, the velocity increases by 32 ft/sec.

Strengthen Your Understanding

61. The line $y = 0.5 - 3x$ has a negative slope and is therefore a decreasing function.

62. If y is directly proportional to x we have $y = kx$. Adding the constant 1 to give $y = 2x + 1$ means that y is not proportional to x.

63. One possible answer is $f(x) = 2x + 3$.

64. One possible answer is $q = \dfrac{8}{p^{1/3}}$.

65. False. A line can be put through any two points in the plane. However, if the line is vertical, it is not the graph of a function.

66. True. Suppose we start at $x = x_1$ and increase x by 1 unit to $x_1 + 1$. If $y = b + mx$, the corresponding values of y are $b + mx_1$ and $b + m(x_1 + 1)$. Thus y increases by

$$\Delta y = b + m(x_1 + 1) - (b + mx_1) = m.$$

67. False. For example, let $y = x + 1$. Then the points $(1, 2)$ and $(2, 3)$ are on the line. However the ratios

$$\frac{2}{1} = 2 \qquad \text{and} \qquad \frac{3}{2} = 1.5$$

are different. The ratio y/x is constant for linear functions of the form $y = mx$, but not in general. (Other examples are possible.)

68. False. For example, if $y = 4x+1$ (so $m = 4$) and $x = 1$, then $y = 5$. Increasing x by 2 units gives 3, so $y = 4(3)+1 = 13$. Thus, y has increased by 8 units, not $4 + 2 = 6$. (Other examples are possible.)

69. (b) and (c). For $g(x) = \sqrt{x}$, the domain and range are all nonnegative numbers, and for $h(x) = x^3$, the domain and range are all real numbers.

Solutions for Section 1.2

Exercises

1. The graph shows a concave up function.

2. The graph shows a concave down function.

3. This graph is neither concave up or down.

4. The graph is concave up.

5. Initial quantity $= 5$; growth rate $= 0.07 = 7\%$.

6. Initial quantity $= 7.7$; growth rate $= -0.08 = -8\%$ (decay).

7. Initial quantity $= 3.2$; growth rate $= 0.03 = 3\%$ (continuous).

8. Initial quantity $= 15$; growth rate $= -0.06 = -6\%$ (continuous decay).

9. Since $e^{0.25t} = \left(e^{0.25}\right)^t \approx (1.2840)^t$, we have $P = 15(1.2840)^t$. This is exponential growth since 0.25 is positive. We can also see that this is growth because $1.2840 > 1$.

10. Since $e^{-0.5t} = (e^{-0.5})^t \approx (0.6065)^t$, we have $P = 2(0.6065)^t$. This is exponential decay since -0.5 is negative. We can also see that this is decay because $0.6065 < 1$.

11. $P = P_0(e^{0.2})^t = P_0(1.2214)^t$. Exponential growth because $0.2 > 0$ or $1.2214 > 1$.

12. $P = 7(e^{-\pi})^t = 7(0.0432)^t$. Exponential decay because $-\pi < 0$ or $0.0432 < 1$.

13. (a) Let $Q = Q_0 a^t$. Then $Q_0 a^5 = 75.94$ and $Q_0 a^7 = 170.86$. So
$$\frac{Q_0 a^7}{Q_0 a^5} = \frac{170.86}{75.94} = 2.25 = a^2.$$
So $a = 1.5$.
(b) Since $a = 1.5$, the growth rate is $r = 0.5 = 50\%$.

14. (a) Let $Q = Q_0 a^t$. Then $Q_0 a^{0.02} = 25.02$ and $Q_0 a^{0.05} = 25.06$. So
$$\frac{Q_0 a^{0.05}}{Q_0 a^{0.02}} = \frac{25.06}{25.02} = 1.001 = a^{0.03}.$$
So
$$a = (1.001)^{\frac{100}{3}} = 1.05.$$
(b) Since $a = 1.05$, the growth rate is $r = 0.05 = 5\%$.

15. (a) The function is linear with initial population of 1000 and slope of 50, so $P = 1000 + 50t$.
(b) This function is exponential with initial population of 1000 and growth rate of 5%, so $P = 1000(1.05)^t$.

16. (a) This is a linear function with slope -2 grams per day and intercept 30 grams. The function is $Q = 30 - 2t$, and the graph is shown in Figure 1.10.

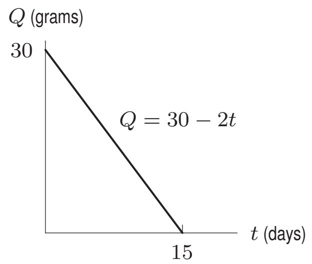


Figure 1.10

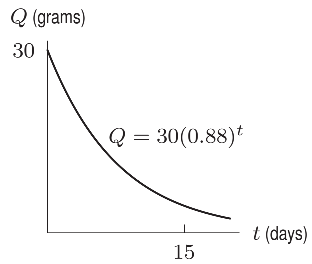


Figure 1.11

(b) Since the quantity is decreasing by a constant percent change, this is an exponential function with base $1 - 0.12 = 0.88$. The function is $Q = 30(0.88)^t$, and the graph is shown in Figure 1.11.

17. The function is increasing and concave up between D and E, and between H and I. It is increasing and concave down between A and B, and between E and F. It is decreasing and concave up between C and D, and between G and H. Finally, it is decreasing and concave down between B and C, and between F and G

18. (a) It was decreasing from March 2 to March 5 and increasing from March 5 to March 9.
(b) From March 5 to 8, the average temperature increased, but the rate of increase went down, from 12° between March 5 and 6 to 4° between March 6 and 7 to 2° between March 7 and 8.
From March 7 to 9, the average temperature increased, and the rate of increase went up, from 2° between March 7 and 8 to 9° between March 8 and 9.

Problems

19. (a) A linear function must change by exactly the same amount whenever x changes by some fixed quantity. While $h(x)$ decreases by 3 whenever x increases by 1, $f(x)$ and $g(x)$ fail this test, since both change by different amounts between $x = -2$ and $x = -1$ and between $x = -1$ and $x = 0$. So the only possible linear function is $h(x)$, so it will be given by a formula of the type: $h(x) = mx + b$. As noted, $m = -3$. Since the y-intercept of h is 31, the formula for $h(x)$ is $h(x) = 31 - 3x$.
(b) An exponential function must grow by exactly the same factor whenever x changes by some fixed quantity. Here, $g(x)$ increases by a factor of 1.5 whenever x increases by 1. Since the y-intercept of $g(x)$ is 36, $g(x)$ has the formula $g(x) = 36(1.5)^x$. The other two functions are not exponential; $h(x)$ is not because it is a linear function, and $f(x)$ is not because it both increases and decreases.

20. Table A and Table B could represent linear functions of x. Table A could represent the constant linear function $y = 2.2$ because all y values are the same. Table B could represent a linear function of x with slope equal to $11/4$. This is because x values that differ by 4 have corresponding y values that differ by 11, and x values that differ by 8 have corresponding y values that differ by 22. In Table C, y decreases and then increases as x increases, so the table cannot represent a linear function. Table D does not show a constant rate of change, so it cannot represent a linear function.

21. Table D is the only table that could represent an exponential function of x. This is because, in Table D, the ratio of y values is the same for all equally spaced x values. Thus, the y values in the table have a constant percent rate of decrease:

$$\frac{9}{18} = \frac{4.5}{9} = \frac{2.25}{4.5} = 0.5.$$

Table A represents a constant function of x, so it cannot represent an exponential function. In Table B, the ratio between y values corresponding to equally spaced x values is not the same. In Table C, y decreases and then increases as x increases. So neither Table B nor Table C can represent exponential functions.

22. (a) Let P represent the population of the world, and let t represent the number of years since 2010. Then we have $P = 6.91(1.011)^t$.

(b) According to this formula, the population of the world in the year 2020 (at $t = 10$) will be $P = 6.9(1.011)^{10} = 7.71$ billion people.

(c) The graph is shown in Figure 1.12. The population of the world has doubled when $P = 13.82$; we see on the graph that this occurs at approximately $t = 63.4$. Under these assumptions, the doubling time of the world's population is about 63.4 years.

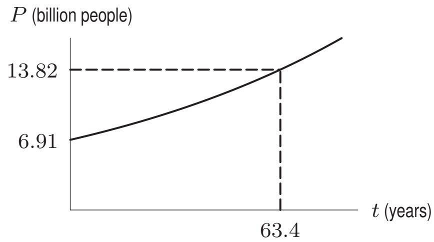
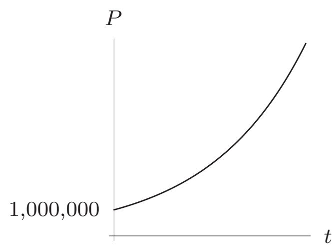


Figure 1.12

23. (a) We have $P_0 = 1$ million, and $k = 0.02$, so $P = (1{,}000{,}000)(e^{0.02t})$.

(b)



24. The doubling time t depends only on the growth rate; it is the solution to

$$2 = (1.02)^t,$$

since 1.02^t represents the factor by which the population has grown after time t. Trial and error shows that $(1.02)^{35} \approx 1.9999$ and $(1.02)^{36} \approx 2.0399$, so that the doubling time is about 35 years.

25. (a) We have

$$\text{Reduced size } = (0.80) \cdot \text{ Original size}$$

or

$$\text{Original size} = \frac{1}{(0.80)} \text{ Reduced size } = (1.25) \text{ Reduced size},$$

so the copy must be enlarged by a factor of 1.25, which means it is enlarged to 125% of the reduced size.

(b) If a page is copied n times, then

$$\text{New size } = (0.80)^n \cdot \text{ Original.}$$

We want to solve for n so that

$$(0.80)^n = 0.15.$$

By trial and error, we find $(0.80)^8 = 0.168$ and $(0.80)^9 = 0.134$. So the page needs to be copied 9 times.

26. (a) See Figure 1.13.

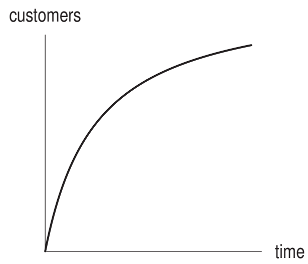


Figure 1.13

(b) "The rate at which new people try it" is the rate of change of the total number of people who have tried the product. Thus, the statement of the problem is telling you that the graph is concave down—the slope is positive but decreasing, as the graph shows.

27. (a) Advertising is generally cheaper in bulk; spending more money will give better and better marginal results initially, (Spending $\$5{,}000$ could give you a big newspaper ad reaching 200,000 people; spending $\$100{,}000$ could give you a series of TV spots reaching 50,000,000 people.) See Figure 1.14.

(b) The temperature of a hot object decreases at a rate proportional to the difference between its temperature and the temperature of the air around it. Thus, the temperature of a very hot object decreases more quickly than a cooler object. The graph is decreasing and concave up. See Figure 1.15 (We are assuming that the coffee is all at the same temperature.)

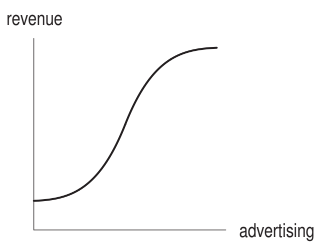


Figure 1.14

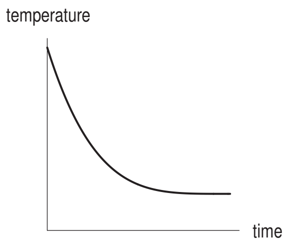


Figure 1.15

28. (a) This is the graph of a linear function, which increases at a constant rate, and thus corresponds to $k(t)$, which increases by 0.3 over each interval of 1.

(b) This graph is concave down, so it corresponds to a function whose increases are getting smaller, as is the case with $h(t)$, whose increases are 10, 9, 8, 7, and 6.

(c) This graph is concave up, so it corresponds to a function whose increases are getting bigger, as is the case with $g(t)$, whose increases are 1, 2, 3, 4, and 5.

29. (a) This is a linear function, corresponding to $g(x)$, whose rate of decrease is constant, 0.6.

(b) This graph is concave down, so it corresponds to a function whose rate of decrease is increasing, like $h(x)$. (The rates are -0.2, -0.3, -0.4, -0.5, -0.6.)

(c) This graph is concave up, so it corresponds to a function whose rate of decrease is decreasing, like $f(x)$. (The rates are -10, -9, -8, -7, -6.)

30. Since we are told that the rate of decay is *continuous*, we use the function $Q(t) = Q_0e^{rt}$ to model the decay, where $Q(t)$ is the amount of strontium-90 which remains at time t, and Q_0 is the original amount. Then

$$Q(t) = Q_0e^{-0.0247t}.$$

So after 100 years,

$$Q(100) = Q_0e^{-0.0247\cdot 100}$$

and

$$\frac{Q(100)}{Q_0} = e^{-2.47} \approx 0.0846$$

so about 8.46% of the strontium-90 remains.

31. We look for an equation of the form $y = y_0 a^x$ since the graph looks exponential. The points $(0, 3)$ and $(2, 12)$ are on the graph, so

$$3 = y_0 a^0 = y_0$$

and

$$12 = y_0 \cdot a^2 = 3 \cdot a^2, \quad \text{giving} \quad a = \pm 2.$$

Since $a > 0$, our equation is $y = 3(2^x)$.

32. We look for an equation of the form $y = y_0 a^x$ since the graph looks exponential. The points $(-1, 8)$ and $(1, 2)$ are on the graph, so

$$8 = y_0 a^{-1} \quad \text{and} \quad 2 = y_0 a^1$$

Therefore $\frac{8}{2} = \frac{y_0 a^{-1}}{y_0 a} = \frac{1}{a^2}$, giving $a = \frac{1}{2}$, and so $2 = y_0 a^1 = y_0 \cdot \frac{1}{2}$, so $y_0 = 4$.
Hence $y = 4\left(\frac{1}{2}\right)^x = 4(2^{-x})$.

33. We look for an equation of the form $y = y_0 a^x$ since the graph looks exponential. The points $(1, 6)$ and $(2, 18)$ are on the graph, so

$$6 = y_0 a^1 \quad \text{and} \quad 18 = y_0 a^2$$

Therefore $a = \frac{y_0 a^2}{y_0 a} = \frac{18}{6} = 3$, and so $6 = y_0 a = y_0 \cdot 3$; thus, $y_0 = 2$. Hence $y = 2(3^x)$.

34. The difference, D, between the horizontal asymptote and the graph appears to decrease exponentially, so we look for an equation of the form

$$D = D_0 a^x$$

where $D_0 = 4 =$ difference when $x = 0$. Since $D = 4 - y$, we have

$$4 - y = 4a^x \quad \text{or} \quad y = 4 - 4a^x = 4(1 - a^x)$$

The point $(1, 2)$ is on the graph, so $2 = 4(1 - a^1)$, giving $a = \frac{1}{2}$.
Therefore $y = 4(1 - (\frac{1}{2})^x) = 4(1 - 2^{-x})$.

35. Since f is linear, its slope is a constant

$$m = \frac{20 - 10}{2 - 0} = 5.$$

Thus f increases 5 units for unit increase in x, so

$$f(1) = 15, \quad f(3) = 25, \quad f(4) = 30.$$

Since g is exponential, its growth factor is constant. Writing $g(x) = ab^x$, we have $g(0) = a = 10$, so

$$g(x) = 10 \cdot b^x.$$

Since $g(2) = 10 \cdot b^2 = 20$, we have $b^2 = 2$ and since $b > 0$, we have

$$b = \sqrt{2}.$$

Thus g increases by a factor of $\sqrt{2}$ for unit increase in x, so

$$g(1) = 10\sqrt{2}, \quad g(3) = 10(\sqrt{2})^3 = 20\sqrt{2}, \quad g(4) = 10(\sqrt{2})^4 = 40.$$

Notice that the value of $g(x)$ doubles between $x = 0$ and $x = 2$ (from $g(0) = 10$ to $g(2) = 20$), so the doubling time of $g(x)$ is 2. Thus, $g(x)$ doubles again between $x = 2$ and $x = 4$, confirming that $g(4) = 40$.

36. We see that $\frac{1.09}{1.06} \approx 1.03$, and therefore $h(s) = c(1.03)^s$; c must be 1. Similarly $\frac{2.42}{2.20} = 1.1$, and so $f(s) = a(1.1)^s$; $a = 2$. Lastly, $\frac{3.65}{3.47} \approx 1.05$, so $g(s) = b(1.05)^s$; $b \approx 3$.

37. (a) Because the population is growing exponentially, the time it takes to double is the same, regardless of the population levels we are considering. For example, the population is 20,000 at time 3.7, and 40,000 at time 6.0. This represents a doubling of the population in a span of $6.0 - 3.7 = 2.3$ years.

How long does it take the population to double a second time, from 40,000 to 80,000? Looking at the graph once again, we see that the population reaches 80,000 at time $t = 8.3$. This second doubling has taken $8.3 - 6.0 = 2.3$ years, the same amount of time as the first doubling.

Further comparison of any two populations on this graph that differ by a factor of two will show that the time that separates them is 2.3 years. Similarly, during any 2.3 year period, the population will double. Thus, the doubling time is 2.3 years.

(b) Suppose $P = P_0 a^t$ doubles from time t to time $t + d$. We now have $P_0 a^{t+d} = 2P_0 a^t$, so $P_0 a^t a^d = 2P_0 a^t$. Thus, canceling P_0 and a^t, d must be the number such that $a^d = 2$, no matter what t is.

38. (a) After 50 years, the amount of money is

$$P = 2P_0.$$

After 100 years, the amount of money is

$$P = 2(2P_0) = 4P_0.$$

After 150 years, the amount of money is

$$P = 2(4P_0) = 8P_0.$$

(b) The amount of money in the account doubles every 50 years. Thus in t years, the balance doubles $t/50$ times, so

$$P = P_0 2^{t/50}.$$

39. (a) Since $162.5 = 325/2$, there are 162.5 mg remaining after H hours.
Since $81.25 = 162.5/2$, there are 81.25 mg remaining H hours after there were 162.5 mg, so $2H$ hours after there were 325 mg.
Since $40.625 = 81.25/2$, there are 41.625 mg remaining H hours after there were 81.25 mg, so $3H$ hours after there were 325 mg.

(b) Each additional H hours, the quantity is halved. Thus in t hours, the quantity was halved t/H times, so

$$A = 325\left(\frac{1}{2}\right)^{t/H}.$$

40. (a) The quantity of radium decays exponentially, so we know $Q = Q_0 a^t$. When $t = 1620$, we have $Q = Q_0/2$ so

$$\frac{Q_0}{2} = Q_0 a^{1620}.$$

Thus, canceling Q_0, we have

$$a^{1620} = \frac{1}{2}$$
$$a = \left(\frac{1}{2}\right)^{1/1620}.$$

Thus the formula is $Q = Q_0\left(\left(\frac{1}{2}\right)^{1/1620}\right)^t = Q_0\left(\frac{1}{2}\right)^{t/1620}$.

(b) After 500 years,

$$\text{Fraction remaining } = \frac{1}{Q_0}\cdot Q_0\left(\frac{1}{2}\right)^{500/1620} = 0.80740.$$

so 80.740% is left.

41. Let Q_0 be the initial quantity absorbed in 1960. Then the quantity, Q, of strontium-90 left after t years is

$$Q = Q_0\left(\frac{1}{2}\right)^{t/29}.$$

Since $2010 - 1960 = 50$ years, in 2010

$$\text{Fraction remaining } = \frac{1}{Q_0}\cdot Q_0\left(\frac{1}{2}\right)^{50/29} = \left(\frac{1}{2}\right)^{50/29} = 0.30268 = 30.268\%.$$

42. Direct calculation reveals that each 1000 foot increase in altitude results in a longer takeoff roll by a factor of about 1.096. Since the value of d when $h = 0$ (sea level) is $d = 670$, we are led to the formula

$$d = 670(1.096)^{h/1000},$$

where d is the takeoff roll, in feet, and h is the airport's elevation, in feet.

Alternatively, we can write

$$d = d_0 a^h,$$

where d_0 is the sea level value of d, $d_0 = 670$. In addition, when $h = 1000$, $d = 734$, so

$$734 = 670a^{1000}.$$

Solving for a gives

$$a = \left(\frac{734}{670}\right)^{1/1000} = 1.00009124,$$

so

$$d = 670(1.00009124)^h.$$

43. (a) Since the annual growth factor from 2005 to 2006 was $1 + 1.866 = 2.866$ and $91(1 + 1.866) = 260.806$, the US consumed approximately 261 million gallons of biodiesel in 2006. Since the annual growth factor from 2006 to 2007 was $1 + 0.372 = 1.372$ and $261(1 + 0.372) = 358.092$, the US consumed about 358 million gallons of biodiesel in 2007.

(b) Completing the table of annual consumption of biodiesel and plotting the data gives Figure 1.16.

Year	2005	2006	2007	2008	2009
Consumption of biodiesel (mn gal)	91	261	358	316	339

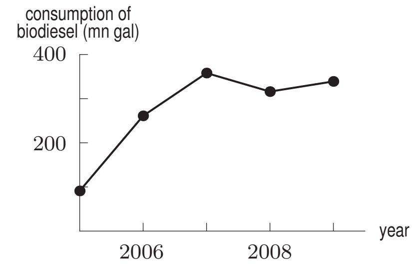


Figure 1.16

44. (a) False, because the annual percent growth is not constant over this interval.

(b) The US consumption of biodiesel more than doubled in 2005 and more than doubled again in 2006. This is because the annual percent growth was larger than 100% for both of these years.

(c) The US consumption of biodiesel more than tripled in 2005, since the annual percent growth in 2005 was over 200%

45. (a) Since the annual growth factor from 2006 to 2007 was $1 - 0.454 = 0.546$ and $29(1 - 0.454) = 15.834$, the US consumed approximately 16 trillion BTUs of hydroelectric power in 2007. Since the annual growth factor from 2005 to 2006 was $1 - 0.10 = 0.90$ and $\frac{29}{(1 - 0.10)} = 32.222$, the US consumed about 32 trillion BTUs of hydroelectric power in 2005.

(b) Completing the table of annual consumption of hydroelectric power and plotting the data gives Figure 1.17.

Year	2004	2005	2006	2007	2008	2009
Consumption of hydro. power (trillion BTU)	33	32	29	16	17	19

(c) The largest decrease in the US consumption of hydroelectric power occurred in 2007. In this year, the US consumption of hydroelectric power dropped by about 13 trillion BTUs to 16 trillion BTUs, down from 29 trillion BTUs in 2006.

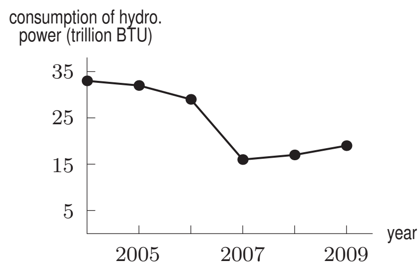


Figure 1.17

46. (a) From the figure we can read-off the approximate percent growth for each year over the previous year:

Year	2005	2006	2007	2008	2009
% growth over previous yr	25	50	30	60	29

Since the annual growth factor from 2006 to 2007 was $1 + 0.30 = 1.30$ and

$$\frac{341}{(1 + 0.30)} = 262.31,$$

the US consumed approximately 262 trillion BTUs of wind power energy in 2006. Since the annual growth factor from 2007 to 2008 was $1 + 0.60 = 1.60$ and $341(1 + 0.60) = 545.6$, the US consumed about 546 trillion BTUs of wind power energy in 2008.

(b) Completing the table of annual consumption of wind power and plotting the data gives Figure 1.18.

Year	2005	2006	2007	2008	2009
Consumption of wind power (trillion BTU)	175	262	341	546	704

(c) The largest increase in the US consumption of wind power energy occurred in 2008. In this year the US consumption of wind power energy rose by about 205 trillion BTUs to 546 trillion BTUs, up from 341 trillion BTUs in 2007.

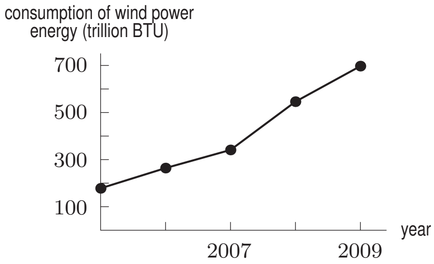


Figure 1.18

47. (a) The US consumption of wind power energy increased by at least 40% in 2006 and in 2008, relative to the previous year. In 2006 consumption increased by just under 50% over consumption in 2005, and in 2008 consumption increased by about 60% over consumption in 2007. Consumption did not decrease during the time period shown because all the annual percent growth values are positive, indicating a steady increase in the US consumption of wind power energy between 2005 and 2009.

(b) Yes. From 2006 to 2007 consumption increased by about 30%, which means $x(1+0.30)$ units of wind power energy were consumed in 2007 if x had been consumed in 2006. Similarly,

$$(x(1 + 0.30))(1 + 0.60)$$

units of wind power energy were consumed in 2008 if x had been consumed in 2006 (because consumption increased by about 60% from 2007 to 2008). Since

$$(x(1 + 0.30))(1 + 0.60) = x(2.08) = x(1 + 1.08),$$

the percent growth in wind power consumption was about 108%, or just over 100%, in 2008 relative to consumption in 2006.

Strengthen Your Understanding

48. The function $y = e^{-0.25x}$ is decreasing but its graph is concave up.

49. The graph of $y = 2x$ is a straight line and is neither concave up or concave down.

50. One possible answer is $q = 2.2(0.97)^t$.

51. One possible answer is $f(x) = 2(1.1)^x$.

52. One possibility is $y = e^{-x} - 5$.

53. False. The y-intercept is $y = 2 + 3e^{-0} = 5$.

54. True, since, as $t \to \infty$, we know $e^{-4t} \to 0$, so $y = 5 - 3e^{-4t} \to 5$.

55. False. Suppose $y = 5^x$. Then increasing x by 1 increases y by a factor of 5. However increasing x by 2 increases y by a factor of 25, not 10, since

$$y = 5^{x+2} = 5^x \cdot 5^2 = 25 \cdot 5^x.$$

(Other examples are possible.)

56. True. Suppose $y = Ab^x$ and we start at the point (x_1, y_1), so $y_1 = Ab^{x_1}$. Then increasing x_1 by 1 gives $x_1 + 1$, so the new y-value, y_2, is given by

$$y_2 = Ab^{x_1+1} = Ab^{x_1}b = (Ab^{x_1})b,$$

so

$$y_2 = by_1.$$

Thus, y has increased by a factor of b, so $b = 3$, and the function is $y = A3^x$.

However, if x_1 is increased by 2, giving $x_1 + 2$, then the new y-value, y_3, is given by

$$y_3 = A3^{x_1+2} = A3^{x_1}3^2 = 9A3^{x_1} = 9y_1.$$

Thus, y has increased by a factor of 9.

57. True. For example, $f(x) = (0.5)^x$ is an exponential function which decreases. (Other examples are possible.)

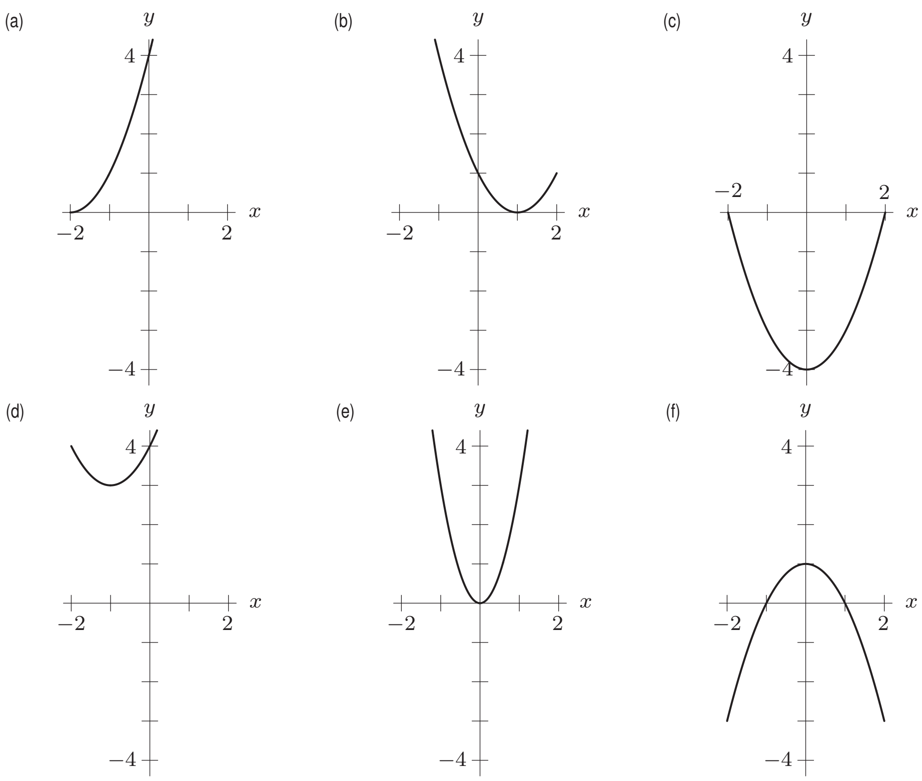
58. True. If $b > 1$, then $ab^x \to 0$ as $x \to -\infty$. If $0 < b < 1$, then $ab^x \to 0$ as $x \to \infty$. In either case, the function $y = a + ab^x$ has $y = a$ as the horizontal asymptote.

59. True, since $e^{-kt} \to 0$ as $t \to \infty$, so $y \to 20$ as $t \to \infty$.

Solutions for Section 1.3

Exercises

1.



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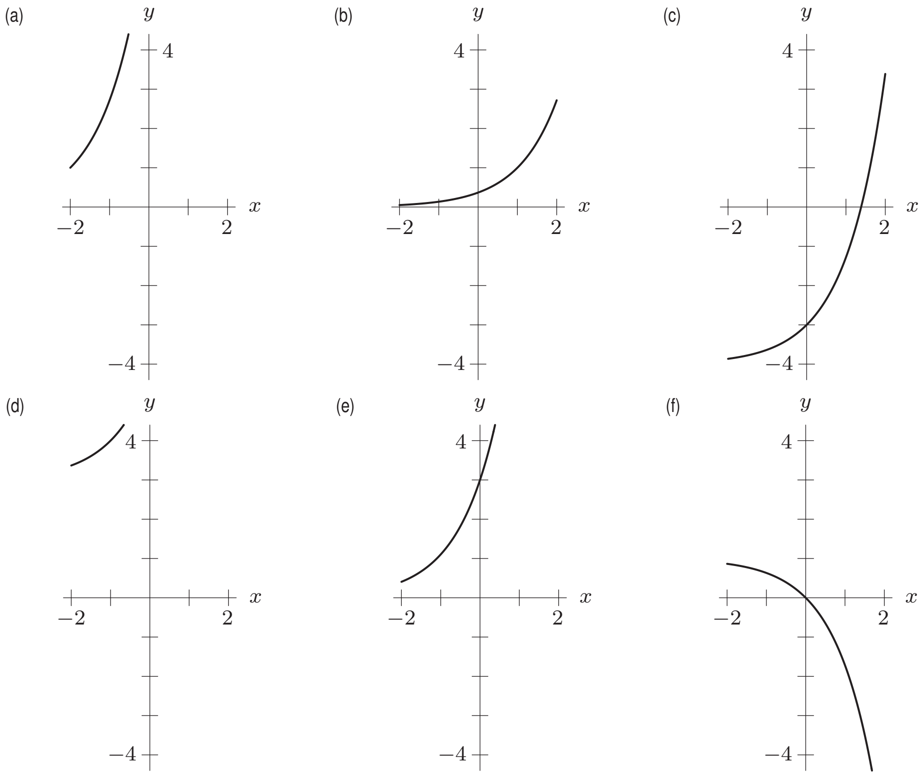

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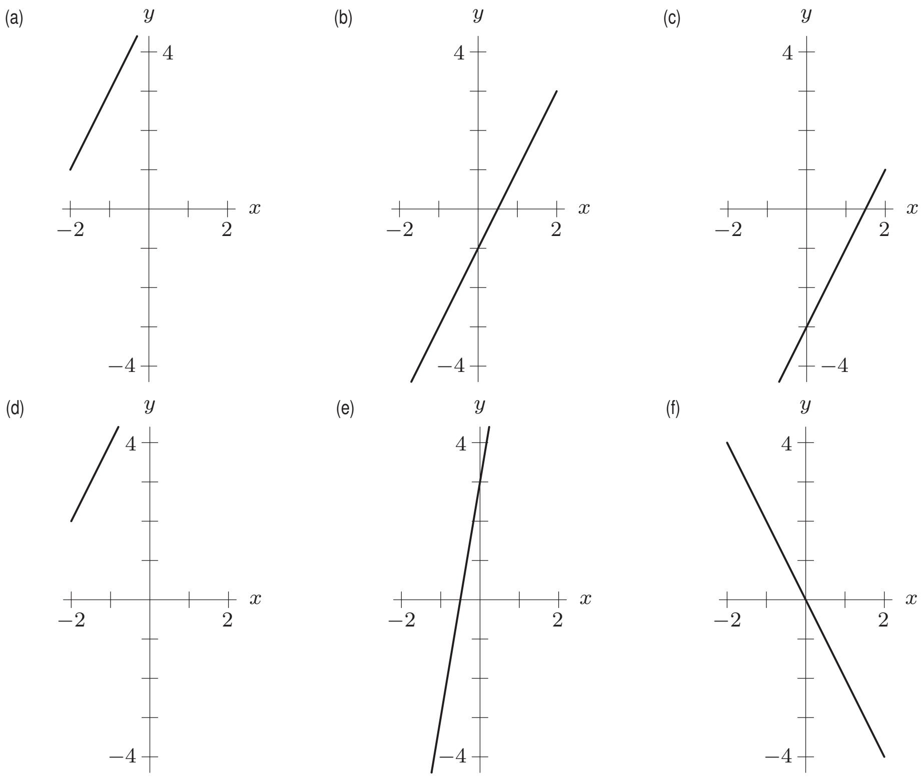


4. This graph is the graph of $m(t)$ shifted upward by two units. See Figure 1.19.

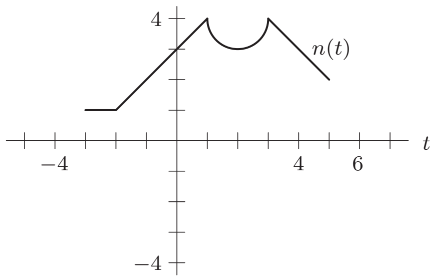


Figure 1.19

5. This graph is the graph of $m(t)$ shifted to the right by one unit. See Figure 1.20.

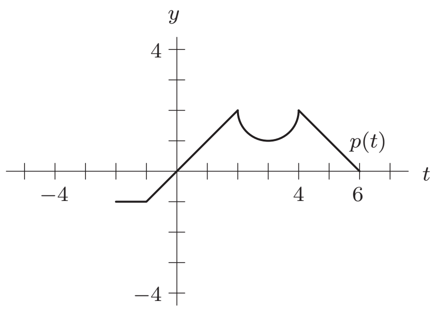


Figure 1.20

6. This graph is the graph of $m(t)$ shifted to the left by 1.5 units. See Figure 1.21.

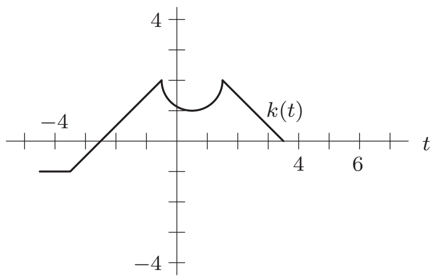


Figure 1.21

7. This graph is the graph of $m(t)$ shifted to the right by 0.5 units and downward by 2.5 units. See Figure 1.22.

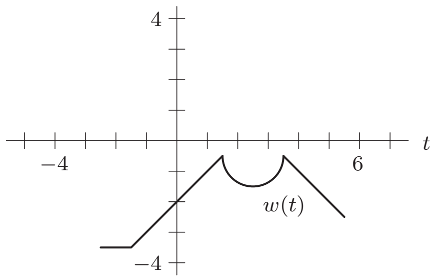


Figure 1.22

8. (a) $f(g(1)) = f(1+1) = f(2) = 2^2 = 4$
(b) $g(f(1)) = g(1^2) = g(1) = 1+1 = 2$
(c) $f(g(x)) = f(x+1) = (x+1)^2$
(d) $g(f(x)) = g(x^2) = x^2+1$
(e) $f(t)g(t) = t^2(t+1)$

9. (a) $f(g(1)) = f(1^2) = f(1) = \sqrt{1+4} = \sqrt{5}$
(b) $g(f(1)) = g(\sqrt{1+4}) = g(\sqrt{5}) = (\sqrt{5})^2 = 5$
(c) $f(g(x)) = f(x^2) = \sqrt{x^2+4}$
(d) $g(f(x)) = g(\sqrt{x+4}) = (\sqrt{x+4})^2 = x+4$
(e) $f(t)g(t) = (\sqrt{t+4})t^2 = t^2\sqrt{t+4}$

10. (a) $f(g(1)) = f(1^2) = f(1) = e^1 = e$
(b) $g(f(1)) = g(e^1) = g(e) = e^2$
(c) $f(g(x)) = f(x^2) = e^{x^2}$
(d) $g(f(x)) = g(e^x) = (e^x)^2 = e^{2x}$
(e) $f(t)g(t) = e^t t^2$

11. (a) $f(g(1)) = f(3\cdot 1+4) = f(7) = \dfrac{1}{7}$
(b) $g(f(1)) = g(1/1) = g(1) = 7$
(c) $f(g(x)) = f(3x+4) = \dfrac{1}{3x+4}$
(d) $g(f(x)) = g\left(\dfrac{1}{x}\right) = 3\left(\dfrac{1}{x}\right)+4 = \dfrac{3}{x}+4$
(e) $f(t)g(t) = \dfrac{1}{t}(3t+4) = 3+\dfrac{4}{t}$

12. (a) $g(2+h) = (2+h)^2 + 2(2+h) + 3 = 4+4h+h^2+4+2h+3 = h^2+6h+11$.
(b) $g(2) = 2^2 + 2(2) + 3 = 4+4+3 = 11$, which agrees with what we get by substituting $h = 0$ into (a).
(c) $g(2+h) - g(2) = (h^2+6h+11) - (11) = h^2+6h$.

13. (a) $f(t+1) = (t+1)^2 + 1 = t^2+2t+1+1 = t^2+2t+2$.
(b) $f(t^2+1) = (t^2+1)^2 + 1 = t^4+2t^2+1+1 = t^4+2t^2+2$.
(c) $f(2) = 2^2+1 = 5$.
(d) $2f(t) = 2(t^2+1) = 2t^2+2$.
(e) $(f(t))^2 + 1 = \left(t^2+1\right)^2 + 1 = t^4+2t^2+1+1 = t^4+2t^2+2$.

14. $m(z+1) - m(z) = (z+1)^2 - z^2 = 2z+1$.

15. $m(z+h) - m(z) = (z+h)^2 - z^2 = 2zh+h^2$.

16. $m(z) - m(z-h) = z^2 - (z-h)^2 = 2zh-h^2$.

17. $m(z+h) - m(z-h) = (z+h)^2 - (z-h)^2 = z^2+2hz+h^2-(z^2-2hz+h^2) = 4hz$.

18. (a) $f(25)$ is q corresponding to $p = 25$, or, in other words, the number of items sold when the price is 25.
(b) $f^{-1}(30)$ is p corresponding to $q = 30$, or the price at which 30 units will be sold.

19. (a) $f(10{,}000)$ represents the value of C corresponding to $A = 10{,}000$, or in other words the cost of building a 10,000 square-foot store.
(b) $f^{-1}(20{,}000)$ represents the value of A corresponding to $C = 20{,}000$, or the area in square feet of a store which would cost \$20,000 to build.

20. $f^{-1}(75)$ is the length of the column of mercury in the thermometer when the temperature is 75°F.

21. (a) The equation is $y = 2x^2+1$. Note that its graph is narrower than the graph of $y = x^2$ which appears in gray. See Figure 1.23.

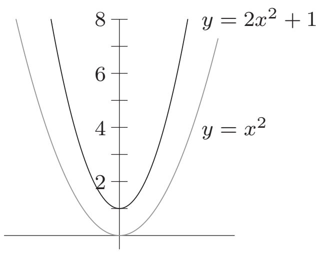


Figure 1.23

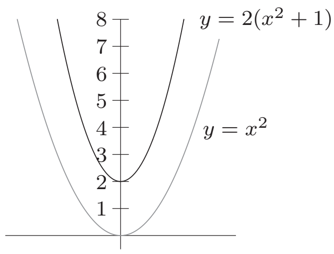


Figure 1.24

(b) $y = 2(x^2 + 1)$ moves the graph up one unit and *then* stretches it by a factor of two. See Figure 1.24.
(c) No, the graphs are not the same. Since $2(x^2 + 1) = (2x^2 + 1) + 1$, the second graph is always one unit higher than the first.

22. Figure 1.25 shows the appropriate graphs. Note that asymptotes are shown as dashed lines and x- or y-intercepts are shown as filled circles.

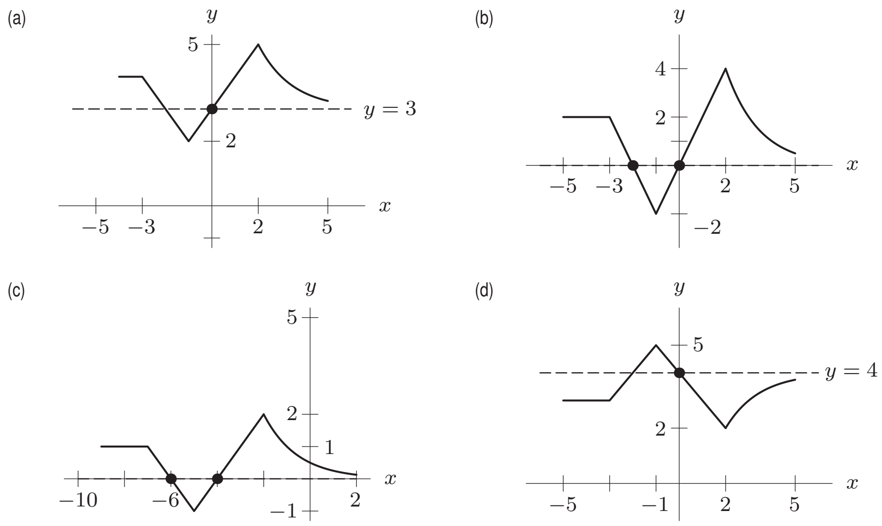


Figure 1.25

23. The function is not invertible since there are many horizontal lines which hit the function twice.

24. The function is not invertible since there are horizontal lines which hit the function more than once.

25. Since a horizontal line cuts the graph of $f(x) = x^2 + 3x + 2$ two times, f is not invertible. See Figure 1.26.

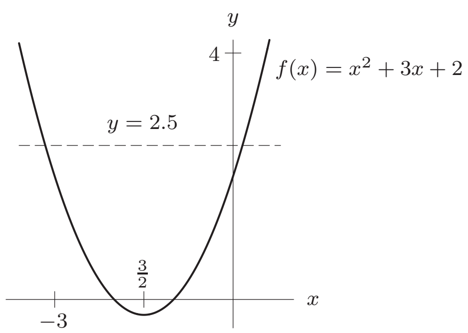


Figure 1.26

26. Since a horizontal line cuts the graph of $f(x) = x^3 - 5x + 10$ three times, f is not invertible. See Figure 1.27.

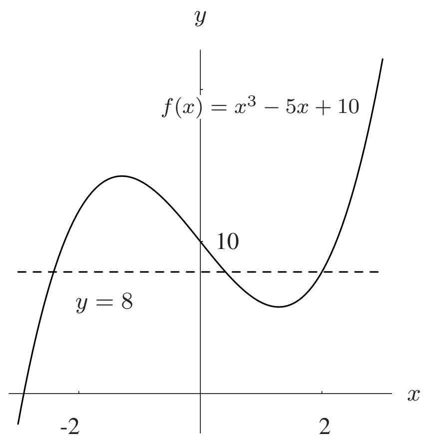


Figure 1.27

27. Since any horizontal line cuts the graph once, f is invertible. See Figure 1.28.

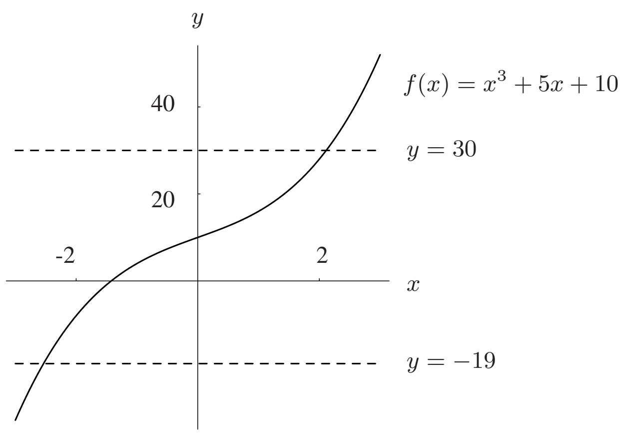


Figure 1.28

28.

$$f(-x) = (-x)^6 + (-x)^3 + 1 = x^6 - x^3 + 1.$$

Since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, this function is neither even nor odd.

29.

$$f(-x) = (-x)^3 + (-x)^2 + (-x) = -x^3 + x^2 - x.$$

Since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, this function is neither even nor odd.

30. Since

$$f(-x) = (-x)^4 - (-x)^2 + 3 = x^4 - x^2 + 3 = f(x),$$

we see f is even

31. Since

$$f(-x) = (-x)^3 + 1 = -x^3 + 1,$$

we see $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, so f is neither even nor odd

32. Since

$$f(-x) = 2(-x) = -2x = -f(x),$$

we see f is odd.

33. Since

$$f(-x) = e^{(-x)^2-1} = e^{x^2-1} = f(x),$$

we see f is even.

34. Since

$$f(-x) = (-x)((-x)^2 - 1) = -x(x^2 - 1) = -f(x),$$

we see f is odd

35. Since

$$f(-x) = e^{-x} + x,$$

we see $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, so f is neither even nor odd

Problems

36. $f(x) = x^3, \quad g(x) = x + 1.$

37. $f(x) = x + 1, \quad g(x) = x^3.$

38. $f(x) = \sqrt{x}, \quad g(x) = x^2 + 4$

39. $f(x) = e^x, \quad g(x) = 2x$

40. This looks like a shift of the graph $y = -x^2$. The graph is shifted to the left 1 unit and up 3 units, so a possible formula is $y = -(x + 1)^2 + 3$.

41. This looks like a shift of the graph $y = x^3$. The graph is shifted to the right 2 units and down 1 unit, so a possible formula is $y = (x - 2)^3 - 1$.

42. (a) We find $f^{-1}(2)$ by finding the x value corresponding to $f(x) = 2$. Looking at the graph, we see that $f^{-1}(2) = -1$.
(b) We construct the graph of $f^{-1}(x)$ by reflecting the graph of $f(x)$ over the line $y = x$. The graphs of $f^{-1}(x)$ and $f(x)$ are shown together in Figure 1.29.

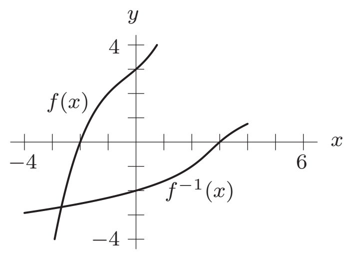


Figure 1.29

43. Values of f^{-1} are as follows

x	3	−7	19	4	178	2	1
$f^{-1}(x)$	1	2	3	4	5	6	7

The domain of f^{-1} is the set consisting of the integers $\{3, -7, 19, 4, 178, 2, 1\}$.

44. f is an increasing function since the amount of fuel used increases as flight time increases. Therefore f is invertible.

45. Not invertible. Given a certain number of customers, say $f(t) = 1500$, there could be many times, t, during the day at which that many people were in the store. So we don't know which time instant is the right one.

46. Probably not invertible. Since your calculus class probably has less than 363 students, there will be at least two days in the year, say a and b, with $f(a) = f(b) = 0$. Hence we don't know what to choose for $f^{-1}(0)$.

47. Not invertible, since it costs the same to mail a 50-gram letter as it does to mail a 51-gram letter.

48. The volume of the balloon t minutes after inflation began is: $g(f(t))$ ft^3.

49. The volume of the balloon if its radius were twice as big is: $g(2r)$ ft^3.

50. The time elapsed is: $f^{-1}(30)$ min.

51. The time elapsed is: $f^{-1}(g^{-1}(10{,}000))$ min.

52. We have $v(10) = 65$ but the graph of u only enables us to evaluate $u(x)$ for $0 \le x \le 50$. There is not enough information to evaluate $u(v(10))$.

53. We have approximately $v(40) = 15$ and $u(15) = 18$ so $u(v(40)) = 18$.

54. We have approximately $u(10) = 13$ and $v(13) = 60$ so $v(u(10)) = 60$.

55. We have $u(40) = 60$ but the graph of v only enables us to evaluate $v(x)$ for $0 \le x \le 50$. There is not enough information to evaluate $v(u(40))$.

56. (a) Yes, f is invertible, since f is increasing everywhere.
(b) The number $f^{-1}(400)$ is the year in which 400 million motor vehicles were registered in the world. From the picture, we see that $f^{-1}(400)$ is around 1979.
(c) Since the graph of f^{-1} is the reflection of the graph of f over the line $y = x$, we get Figure 1.30.

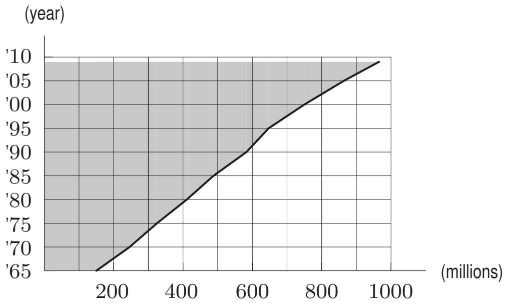


Figure 1.30: Graph of f^{-1}

57. $f(g(1)) = f(2) \approx 0.4$.

58. $g(f(2)) \approx g(0.4) \approx 1.1$.

59. $f(f(1)) \approx f(-0.4) \approx -0.9$.

60. Computing $f(g(x))$ as in Problem 57, we get Table 1.2. From it we graph $f(g(x))$ in Figure 1.31.

Table 1.2

x	$g(x)$	$f(g(x))$
-3	0.6	-0.5
-2.5	-1.1	-1.3
-2	-1.9	-1.2
-1.5	-1.9	-1.2
-1	-1.4	-1.3
-0.5	-0.5	-1
0	0.5	-0.6
0.5	1.4	-0.2
1	2	0.4
1.5	2.2	0.5
2	1.6	0
2.5	0.1	-0.7
3	-2.5	0.1

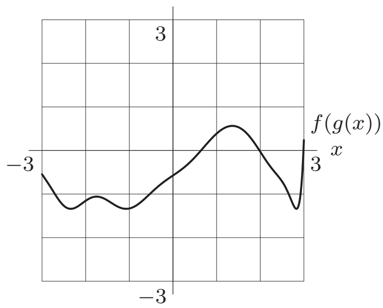


Figure 1.31

61. Using the same way to compute $g(f(x))$ as in Problem 58, we get Table 1.3. Then we can plot the graph of $g(f(x))$ in Figure 1.32.

Table 1.3

x	$f(x)$	$g(f(x))$
-3	3	-2.6
-2.5	0.1	0.8
-2	-1	-1.4
-1.5	-1.3	-1.8
-1	-1.2	-1.7
-0.5	-1	-1.4
0	-0.8	-1
0.5	-0.6	-0.6
1	-0.4	-0.3
1.5	-0.1	0.3
2	0.3	1.1
2.5	0.9	2
3	1.6	2.2

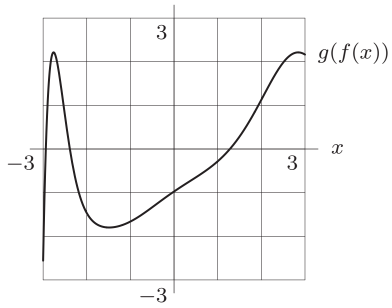


Figure 1.32

62. Using the same way to compute $f(f(x))$ as in Problem 59, we get Table 1.4. Then we can plot the graph of $f(f(x))$ in Figure 1.33.

Table 1.4

x	$f(x)$	$f(f(x))$
-3	3	1.6
-2.5	0.1	-0.7
-2	-1	-1.2
-1.5	-1.3	-1.3
-1	-1.2	-1.3
-0.5	-1	-1.2
0	-0.8	-1.1
0.5	-0.6	-1
1	-0.4	-0.9
1.5	-0.1	-0.8
2	0.3	-0.6
2.5	0.9	-0.4
3	1.6	0

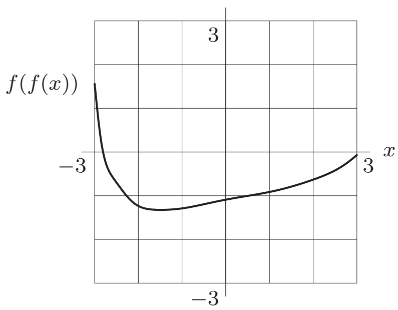


Figure 1.33

63. (a) The graph shows that $f(15)$ is approximately 48. So, the place to find find 15 million-year-old rock is about 48 meters below the Atlantic sea floor.

(b) Since f is increasing (not decreasing, since the depth axis is reversed!), f is invertible. To confirm, notice that the graph of f is cut by a horizontal line at most once.

(c) Look at where the horizontal line through 120 intersects the graph of f and read downward: $f^{-1}(120)$ is about 35. In practical terms, this means that at a depth of 120 meters down, the rock is 35 million years old.

(d) First, we standardize the graph of f so that time and depth are increasing from left to right and bottom to top. Points (t, d) on the graph of f correspond to points (d, t) on the graph of f^{-1}. We can graph f^{-1} by taking points from the original graph of f, reversing their coordinates, and connecting them. This amounts to interchanging the t and d axes, thereby reflecting the graph of f about the line bisecting the 90° angle at the origin. Figure 1.34 is the graph of f^{-1}. (Note that we cannot find the graph of f^{-1} by flipping the graph of f about the line $t = d$ in because t and d have different scales in this instance.)

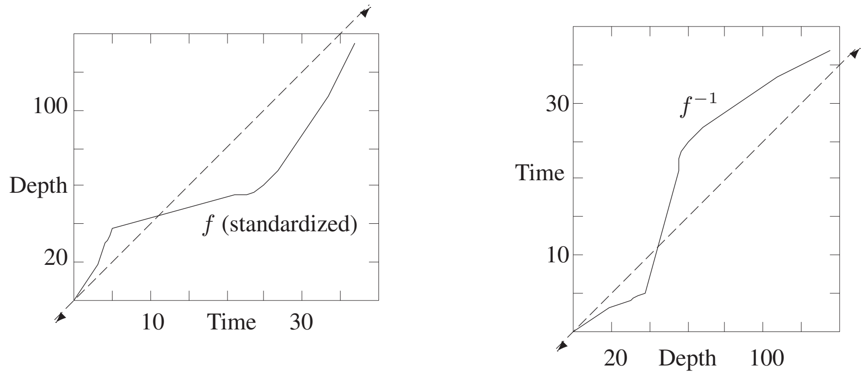


Figure 1.34: Graph of f, reflected to give that of f^{-1}

64. The tree has $B = y - 1$ branches on average and each branch has $n = 2B^2 - B = 2(y-1)^2 - (y-1)$ leaves on average. Therefore

$$\text{Average number of leaves } = Bn = (y-1)(2(y-1)^2 - (y-1)) = 2(y-1)^3 - (y-1)^2.$$

65. The volume, V, of the balloon is $V = \frac{4}{3}\pi r^3$. When $t = 3$, the radius is 10 cm. The volume is then

$$V = \frac{4}{3}\pi(10^3) = \frac{4000\pi}{3} \text{ cm}^3.$$

66. (a) The function f tells us C in terms of q. To get its inverse, we want q in terms of C, which we find by solving for q:

$$\begin{aligned} C &= 100 + 2q, \\ C - 100 &= 2q, \\ q &= \frac{C - 100}{2} = f^{-1}(C). \end{aligned}$$

(b) The inverse function tells us the number of articles that can be produced for a given cost.

67. Since $Q = S - Se^{-kt}$, the graph of Q is the reflection of y about the t-axis moved up by S units.

68.

x	$f(x)$	$g(x)$	$h(x)$
-3	0	0	0
-2	2	2	-2
-1	2	2	-2
0	0	0	0
1	2	-2	-2
2	2	-2	-2
3	0	0	0

Strengthen Your Understanding

69. The graph of $f(x) = -(x+1)^3$ is the graph of $g(x) = -x^3$ shifted left by 1 unit.

70. Since $f(g(x)) = 3(-3x - 5) + 5 = -9x - 10$, we see that f and g are not inverse functions.

71. While $y = 1/x$ is sometimes referred to as the *multiplicative* inverse of x, the inverse of f is $f^{-1}(x) = x$.

72. One possible answer is $g(x) = 3 + x$. (There are many answers.)

73. One possibility is $f(x) = x^2 + 2$.

74. Let $f(x) = 3x$, then $f^{-1}(x) = x/3$. Then for $x > 0$, we have $f(x) > f^{-1}(x)$.

75. We have

$$g(x) = f(x+2)$$

because the graph of g is obtained by moving the graph of f to the left by 2 units. We also have

$$g(x) = f(x) + 3$$

because the graph of g is obtained by moving the graph of f up by 3 units. Thus, we have $f(x+2) = f(x)+3$. The graph of f climbs 3 units whenever x increases by 2. The simplest choice for f is a linear function of slope $3/2$, for example $f(x) = 1.5x$, so $g(x) = 1.5x + 3$.

76. True. The graph of $y = 10^x$ is moved horizontally by h units if we replace x by $x - h$ for some number h. Writing $100 = 10^2$, we have $f(x) = 100(10^x) = 10^2 \cdot 10^x = 10^{x+2}$. The graph of $f(x) = 10^{x+2}$ is the graph of $g(x) = 10^x$ shifted two units to the left.

77. True. If f is increasing then its reflection about the line $y = x$ is also increasing. An example is shown in Figure 1.35. The statement is true.

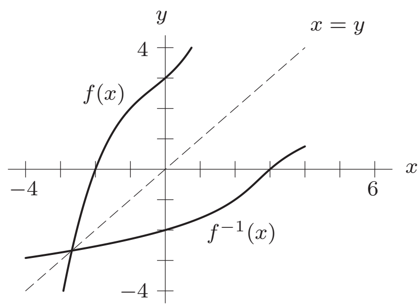


Figure 1.35

78. True. If $f(x)$ is even, we have $f(x) = f(-x)$ for all x. For example, $f(-2) = f(2)$. This means that the graph of $f(x)$ intersects the horizontal line $y = f(2)$ at two points, $x = 2$ and $x = -2$. Thus, f has no inverse function.

79. False. For example, $f(x) = x$ and $g(x) = x^3$ are both odd. Their inverses are $f^{-1}(x) = x$ and $g^{-1}(x) = x^{1/3}$.

80. False. For $x < 0$, as x increases, x^2 decreases, so e^{-x^2} increases.

81. True. We have $g(-x) = g(x)$ since g is even, and therefore $f(g(-x)) = f(g(x))$.

82. False. A counterexample is given by $f(x) = x^2$ and $g(x) = x + 1$. The function $f(g(x)) = (x+1)^2$ is not even because $f(g(1)) = 4$ and $f(g(-1)) = 0 \neq 4$.

83. True. The constant function $f(x) = 0$ is the only function that is both even and odd. This follows, since if f is both even and odd, then, for all x, $f(-x) = f(x)$ (if f is even) and $f(-x) = -f(x)$ (if f is odd). Thus, for all x, $f(x) = -f(x)$ i.e. $f(x) = 0$, for all x. So $f(x) = 0$ is both even and odd and is the only such function.

84. Let $f(x) = x$ and $g(x) = -2x$. Then $f(x) + g(x) = -x$, which is decreasing. Note f is increasing since it has positive slope, and g is decreasing since it has negative slope.

85. This is impossible. If $a < b$, then $f(a) < f(b)$, since f is increasing, and $g(a) > g(b)$, since g is decreasing, so $-g(a) < -g(b)$. Therefore, if $a < b$, then $f(a) - g(a) < f(b) - g(b)$, which means that $f(x) + g(x)$ is increasing.

86. Let $f(x) = e^x$ and let $g(x) = e^{-2x}$. Note f is increasing since it is an exponential growth function, and g is decreasing since it is an exponential decay function. Then $f(x)g(x) = e^{-x}$, which is decreasing.

87. This is impossible. As x increases, $g(x)$ decreases. As $g(x)$ decreases, so does $f(g(x))$ because f is increasing (an increasing function increases as its variable increases, so it decreases as its variable decreases).

Solutions for Section 1.4

Exercises

1. Using the identity $e^{\ln x} = x$, we have $e^{\ln(1/2)} = \frac{1}{2}$.

2. Using the identity $10^{\log x} = x$, we have

$$10^{\log(AB)} = AB$$

3. Using the identity $e^{\ln x} = x$, we have $5A^2$.

4. Using the identity $\ln(e^x) = x$, we have 2AB.

5. Using the rules for ln, we have

$$\begin{aligned}\ln\left(\frac{1}{e}\right) + \ln AB &= \ln 1 - \ln e + \ln A + \ln B \\ &= 0 - 1 + \ln A + \ln B \\ &= -1 + \ln A + \ln B.\end{aligned}$$

6. Using the rules for ln, we have $2A + 3e\ln B$.

7. Taking logs of both sides

$$\begin{aligned}\log 3^x &= x\log 3 = \log 11 \\ x &= \frac{\log 11}{\log 3} = 2.2.\end{aligned}$$

8. Taking logs of both sides

$$\begin{aligned}\log 17^x &= \log 2 \\ x\log 17 &= \log 2 \\ x &= \frac{\log 2}{\log 17} \approx 0.24.\end{aligned}$$

9. Isolating the exponential term

$$\begin{aligned}20 &= 50(1.04)^x \\ \frac{20}{50} &= (1.04)^x.\end{aligned}$$

Taking logs of both sides

$$\begin{aligned}\log\frac{2}{5} &= \log(1.04)^x \\ \log\frac{2}{5} &= x\log(1.04) \\ x &= \frac{\log(2/5)}{\log(1.04)} = -23.4.\end{aligned}$$

10.

$$\begin{aligned}\frac{4}{7} &= \frac{5^x}{3^x} \\ \frac{4}{7} &= \left(\frac{5}{3}\right)^x\end{aligned}$$

Taking logs of both sides

$$\begin{aligned}\log\left(\frac{4}{7}\right) &= x\log\left(\frac{5}{3}\right) \\ x &= \frac{\log(4/7)}{\log(5/3)} \approx -1.1.\end{aligned}$$

11. To solve for x, we first divide both sides by 5 and then take the natural logarithm of both sides.

$$\begin{aligned}\frac{7}{5} &= e^{0.2x}\\ \ln(7/5) &= 0.2x\\ x &= \frac{\ln(7/5)}{0.2} \approx 1.68.\end{aligned}$$

12.
$$\begin{aligned}\ln(2^x) &= \ln(e^{x+1})\\ x\ln 2 &= (x+1)\ln e\\ x\ln 2 &= x+1\\ 0.693x &= x+1\\ x &= \frac{1}{0.693-1} \approx -3.26\end{aligned}$$

13. To solve for x, we first divide both sides by 600 and then take the natural logarithm of both sides.

$$\begin{aligned}\frac{50}{600} &= e^{-0.4x}\\ \ln(50/600) &= -0.4x\\ x &= \frac{\ln(50/600)}{-0.4} \approx 6.212.\end{aligned}$$

14.
$$\begin{aligned}\ln(2e^{3x}) &= \ln(4e^{5x})\\ \ln 2 + \ln(e^{3x}) &= \ln 4 + \ln(e^{5x})\\ 0.693 + 3x &= 1.386 + 5x\\ x &= -0.347\end{aligned}$$

15. Using the rules for ln, we get

$$\begin{aligned}\ln 7^{x+2} &= \ln e^{17x}\\ (x+2)\ln 7 &= 17x\\ x(\ln 7 - 17) &= -2\ln 7\\ x &= \frac{-2\ln 7}{\ln 7 - 17} \approx 0.26.\end{aligned}$$

16.
$$\begin{aligned}\ln(10^{x+3}) &= \ln(5e^{7-x})\\ (x+3)\ln 10 &= \ln 5 + (7-x)\ln e\\ 2.303(x+3) &= 1.609 + (7-x)\\ 3.303x &= 1.609 + 7 - 2.303(3)\\ x &= 0.515\end{aligned}$$

17. Using the rules for ln, we have

$$\begin{aligned}2x - 1 &= x^2\\ x^2 - 2x + 1 &= 0\\ (x-1)^2 &= 0\\ x &= 1.\end{aligned}$$

18.
$$\begin{aligned}4e^{2x-3} &= e+5\\ \ln 4 + \ln(e^{2x-3}) &= \ln(e+5)\\ 1.3863 + 2x - 3 &= 2.0436\\ x &= 1.839.\end{aligned}$$

19. $t = \dfrac{\log a}{\log b}$.

20. $t = \dfrac{\log\left(\frac{P}{P_0}\right)}{\log a} = \dfrac{\log P - \log P_0}{\log a}$.

21. Taking logs of both sides yields

$$nt = \frac{\log\left(\frac{Q}{Q_0}\right)}{\log a}.$$

Hence

$$t = \frac{\log\left(\frac{Q}{Q_0}\right)}{n\log a} = \frac{\log Q - \log Q_0}{n\log a}.$$

22. Collecting similar terms yields

$$\left(\frac{a}{b}\right)^t = \frac{Q_0}{P_0}.$$

Hence

$$t = \frac{\log\left(\frac{Q_0}{P_0}\right)}{\log\left(\frac{a}{b}\right)}.$$

23. $t = \ln\dfrac{a}{b}$.

24. $\ln\dfrac{P}{P_0} = kt$, so $t = \dfrac{\ln\frac{P}{P_0}}{k}$.

25. Since we want $(1.5)^t = e^{kt} = (e^k)^t$, so $1.5 = e^k$, and $k = \ln 1.5 = 0.4055$. Thus, $P = 15e^{0.4055t}$. Since 0.4055 is positive, this is exponential growth.

26. We want $1.7^t = e^{kt}$ so $1.7 = e^k$ and $k = \ln 1.7 = 0.5306$. Thus $P = 10e^{0.5306t}$.

27. We want $0.9^t = e^{kt}$ so $0.9 = e^k$ and $k = \ln 0.9 = -0.1054$. Thus $P = 174e^{-0.1054t}$.

28. Since we want $(0.55)^t = e^{kt} = (e^k)^t$, so $0.55 = e^k$, and $k = \ln 0.55 = -0.5978$. Thus $P = 4e^{-0.5978t}$. Since -0.5978 is negative, this represents exponential decay.

29. If $p(t) = (1.04)^t$, then, for p^{-1} the inverse of p, we should have

$$\begin{aligned}(1.04)^{p^{-1}(t)} &= t,\\ p^{-1}(t)\log(1.04) &= \log t,\\ p^{-1}(t) &= \frac{\log t}{\log(1.04)} \approx 58.708\log t.\end{aligned}$$

30. Since f is increasing, f has an inverse. To find the inverse of $f(t) = 50e^{0.1t}$, we replace t with $f^{-1}(t)$, and, since $f(f^{-1}(t)) = t$, we have

$$t = 50e^{0.1f^{-1}(t)}.$$

We then solve for $f^{-1}(t)$:

$$\begin{aligned}t &= 50e^{0.1f^{-1}(t)}\\ \frac{t}{50} &= e^{0.1f^{-1}(t)}\\ \ln\left(\frac{t}{50}\right) &= 0.1f^{-1}(t)\\ f^{-1}(t) &= \frac{1}{0.1}\ln\left(\frac{t}{50}\right) = 10\ln\left(\frac{t}{50}\right).\end{aligned}$$

31. Using $f(f^{-1}(t)) = t$, we see

$$f(f^{-1}(t)) = 1 + \ln f^{-1}(t) = t.$$

So

$$\begin{aligned}\ln f^{-1}(t) &= t - 1\\ f^{-1}(t) &= e^{t-1}.\end{aligned}$$

Problems

32. The population has increased by a factor of $48{,}000{,}000/40{,}000{,}000 = 1.2$ in 10 years. Thus we have the formula

$$P = 40{,}000{,}000(1.2)^{t/10},$$

and $t/10$ gives the number of 10-year periods that have passed since 2000.

In 2000, $t/10 = 0$, so we have $P = 40{,}000{,}000$.

In 2010, $t/10 = 1$, so $P = 40{,}000{,}000(1.2) = 48{,}000{,}000$.

In 2020, $t/10 = 2$, so $P = 40{,}000{,}000(1.2)^2 = 57{,}600{,}000$.

To find the doubling time, solve $80{,}000{,}000 = 40{,}000{,}000(1.2)^{t/10}$, to get $t = 38.02$ years.

33. In ten years, the substance has decayed to 40% of its original mass. In another ten years, it will decay by an additional factor of 40%, so the amount remaining after 20 years will be $100 \cdot 40\% \cdot 40\% = 16$ kg.

34. We can solve for the growth rate k of the bacteria using the formula $P = P_0 e^{kt}$:

$$\begin{aligned} 1500 &= 500e^{k(2)} \\ k &= \frac{\ln(1500/500)}{2}. \end{aligned}$$

Knowing the growth rate, we can find the population P at time $t = 6$:

$$\begin{aligned} P &= 500e^{(\frac{\ln 3}{2})6} \\ &\approx 13{,}500 \text{ bacteria.} \end{aligned}$$

35. (a) Assuming the US population grows exponentially, we have population $P(t) = 281.4e^{kt}$ at time t years after 2000. Using the 2010 population, we have

$$\begin{aligned} 308.7 &= 281.4e^{10k} \\ k &= \frac{\ln(308.7/281.4)}{10} = 0.00926. \end{aligned}$$

We want to find the time t in which

$$\begin{aligned} 350 &= 281.4e^{0.00926t} \\ t &= \frac{\ln(350/281.4)}{0.00926} = 23.56 \text{ years.} \end{aligned}$$

This model predicts the population to go over 350 million 23.56 years after 2000, in the year 2023.

(b) Evaluate $P = 281.4e^{0.00926t}$ for $t = 20$ to find $P = 338.65$ million people.

36. If C_0 is the concentration of NO_2 on the road, then the concentration x meters from the road is

$$C = C_0 e^{-0.0254x}.$$

We want to find the value of x making $C = C_0/2$, that is,

$$C_0 e^{-0.0254x} = \frac{C_0}{2}.$$

Dividing by C_0 and then taking natural logs yields

$$\ln\left(e^{-0.254x}\right) = -0.0254x = \ln\left(\frac{1}{2}\right) = -0.6931,$$

so

$$x = 27 \text{ meters.}$$

At 27 meters from the road the concentration of NO_2 in the air is half the concentration on the road.

37. (a) Since the percent increase in deaths during a year is constant for constant increase in pollution, the number of deaths per year is an exponential function of the quantity of pollution. If Q_0 is the number of deaths per year without pollution, then the number of deaths per year, Q, when the quantity of pollution is x micrograms per cu meter of air is

$$Q = Q_0(1.0033)^x.$$

(b) We want to find the value of x making $Q = 2Q_0$, that is,

$$Q_0(1.0033)^x = 2Q_0.$$

Dividing by Q_0 and then taking natural logs yields

$$\ln\left((1.0033)^x\right) = x \ln 1.0033 = \ln 2,$$

so

$$x = \frac{\ln 2}{\ln 1.0033} = 210.391.$$

When there are 210.391 micrograms of pollutants per cu meter of air, respiratory deaths per year are double what they would be in the absence of air pollution.

38. (a) Since there are 4 years between 2004 and 2008 we let t be the number of years since 2004 and get:

$$450{,}327 = 211{,}800e^{r4}.$$

Solving for r, we get

$$\begin{aligned}\frac{450{,}327}{211{,}800} &= e^{r4}\\ \ln\left(\frac{450{,}327}{211{,}800}\right) &= 4r\\ r &= 0.188583\end{aligned}$$

Substituting $t = 1, 2, 3$ into

$$211{,}800\, e^{(0.188583)t},$$

we find the three remaining table values:

Year	2004	2005	2006	2007	2008
Number of E85 vehicles	211,800	255,756	308,835	372,930	450,327

(b) If N is the number of E85-powered vehicles in 2003, then

$$211{,}800 = Ne^{0.188583}$$

or

$$N = \frac{211{,}800}{e^{0.188583}} = 175{,}398 \text{ vehicles}.$$

(c) From the table, we can see that the number of E85 vehicles slightly more than doubled from 2004 to 2008, so the percent growth between these years should be slightly over 100%:

$$\begin{array}{c}\text{Percent growth from}\\ \text{2004 to 2008}\end{array} = 100\left(\frac{450{,}327}{211{,}800} - 1\right) = 1.12619 = 112.619\%.$$

39. (a) The initial dose is 10 mg.
(b) Since $0.82 = 1 - 0.18$, the decay rate is 0.18, so 18% leaves the body each hour.
(c) When $t = 6$, we have $A = 10(0.82)^6 = 3.04$. The amount in the body after 6 hours is 3.04 mg.
(d) We want to find the value of t when $A = 1$. Using logarithms:

$$\begin{aligned}1 &= 10(0.82)^t\\ 0.1 &= (0.82)^t\\ \ln(0.1) &= t\ln(0.82)\\ t &= 11.60 \text{ hours}.\end{aligned}$$

After 11.60 hours, the amount is 1 mg.

40. (a) Since the initial amount of caffeine is 100 mg and the exponential decay rate is -0.17, we have $A = 100e^{-0.17t}$.
(b) See Figure 1.36. We estimate the half-life by estimating t when the caffeine is reduced by half (so $A = 50$); this occurs at approximately $t = 4$ hours.

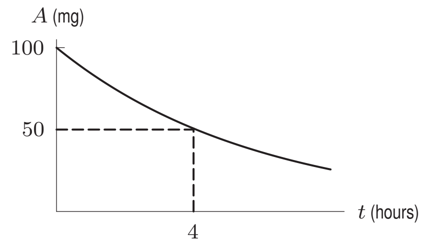


Figure 1.36

(c) We want to find the value of t when $A = 50$:

$$\begin{aligned} 50 &= 100e^{-0.17t} \\ 0.5 &= e^{-0.17t} \\ \ln 0.5 &= -0.17t \\ t &= 4.077. \end{aligned}$$

The half-life of caffeine is about 4.077 hours. This agrees with what we saw in Figure 1.36.

41. Since $y(0) = Ce^0 = C$ we have that $C = 2$. Similarly, substituting $x = 1$ gives $y(1) = 2e^{\alpha}$ so

$$2e^{\alpha} = 1.$$

Rearranging gives $e^{\alpha} = 1/2$. Taking logarithms we get $\alpha = \ln(1/2) = -\ln 2 = -0.693$. Finally,

$$y(2) = 2e^{2(-\ln 2)} = 2e^{-2\ln 2} = \frac{1}{2}.$$

42. The function e^x has a vertical intercept of 1, so must be A. The function $\ln x$ has an x-intercept of 1, so must be D. The graphs of x^2 and $x^{1/2}$ go through the origin. The graph of $x^{1/2}$ is concave down so it corresponds to graph C and the graph of x^2 is concave up so it corresponds to graph B.

43. (a) $B(t) = B_0e^{0.067t}$
(b) $P(t) = P_0e^{0.033t}$
(c) If the initial price is \$50, then

$$\begin{aligned} B(t) &= 50e^{0.067t} \\ P(t) &= 50e^{0.033t}. \end{aligned}$$

We want the value of t such that

$$\begin{aligned} B(t) &= 2P(t) \\ 50e^{0.067t} &= 2 \cdot 50e^{0.033t} \\ \frac{e^{0.067t}}{e^{0.033t}} &= e^{0.034t} = 2 \\ t &= \frac{\ln 2}{0.034} = 20.387 \text{ years}. \end{aligned}$$

Thus, when $t = 20.387$ the price of the textbook was predicted to be double what it would have been had the price risen by inflation only. This occurred in the year 2000.

44. (a) We assume $f(t) = Ae^{-kt}$, where A is the initial population, so $A = 100{,}000$. When $t = 110$, there were 3200 tigers, so

$$3200 = 100{,}000e^{-k\cdot 110}$$

Solving for k gives

$$\begin{aligned} e^{-k\cdot 110} &= \frac{3200}{100{,}000} = 0.0132 \\ k &= -\frac{1}{110}\ln(0.0132) = 0.0313 = 3.13\% \end{aligned}$$

so

$$f(t) = 100{,}000e^{-0.0313t}.$$

(b) In 2000, the predicted number of tigers was

$$f(100) = 100{,}000e^{-0.0313(100)} = 4372.$$

In 2010, we know the number of tigers was 3200. The predicted percent reduction is

$$\frac{3200 - 4372}{4372} = -0.268 = -26.8\%.$$

Thus the actual decrease is larger than the predicted decrease.

45. The population of China, C, in billions, is given by

$$C = 1.34(1.004)^t$$

where t is time measured from 2011, and the population of India, I, in billions, is given by

$$I = 1.19(1.0137)^t.$$

The two populations will be equal when $C = I$, thus, we must solve the equation:

$$1.34(1.004)^t = 1.19(1.0137)^t$$

for t, which leads to

$$\frac{1.34}{1.19} = \frac{(1.0137)^t}{(1.0004)^t} = \left(\frac{1.0137}{1.004}\right)^t.$$

Taking logs on both sides, we get

$$t \log \frac{1.0137}{1.004} = \log \frac{1.34}{1.19},$$

so

$$t = \frac{\log(1.34/1.19)}{\log(1.0137/1.004)} = 12.35 \text{ years.}$$

This model predicts the population of India will exceed that of China in 2023.

46. Let A represent the revenue (in billions of dollars) at Apple t years since 2005. Since $A = 3.68$ when $t = 0$ and we want the continuous growth rate, we write $A = 3.68e^{kt}$. We use the information from 2010, that $A = 15.68$ when $t = 5$, to find k:

$$\begin{aligned} 15.68 &= 3.68e^{k\cdot 5} \\ 4.26 &= e^{5k} \\ \ln(4.26) &= 5k \\ k &= 0.2899. \end{aligned}$$

We have $A = 3.68e^{0.2899t}$, which represents a continuous growth rate of 28.99% per year.

47. Let $P(t)$ be the world population in billions t years after 2010.

(a) Assuming exponential growth, we have

$$P(t) = 6.9e^{kt}.$$

In 2050, we have $t = 40$ and we expect the population then to be 9 billion, so

$$9 = 6.9e^{k\cdot 40}.$$

Solving for k, we have

$$e^{k\cdot 40} = \frac{9}{6.9}$$

$$k = \frac{1}{40}\ln\left(\frac{9}{6.9}\right) = 0.00664 = 0.664\% \text{ per year.}$$

(b) The "Day of 7 Billion" should occur when

$$7 = 6.9e^{0.00664t}.$$

Solving for t gives

$$e^{0.00664t} = \frac{7}{6.9}$$

$$t = \frac{\ln(7/6.9)}{0.00664} = 2.167 \text{ years.}$$

So the "Day of 7 Billion" should be 2.167 years after the end of 2010. This is 2 years and $0.167 \cdot 365 = 61$ days; so 61 days into 2013. That is, March 2, 2013.

48. If r was the average yearly inflation rate, in decimals, then $\frac{1}{4}(1+r)^3 = 2{,}400{,}000$, so $r = 211.53$, i.e. $r = 21{,}153\%$.

49. To find a half-life, we want to find at what t value $Q = \frac{1}{2}Q_0$. Plugging this into the equation of the decay of plutonium-240, we have

$$\begin{aligned}\frac{1}{2} &= e^{-0.00011t}\\ t &= \frac{\ln(1/2)}{-0.00011} \approx 6{,}301 \text{ years.}\end{aligned}$$

The only difference in the case of plutonium-242 is that the constant -0.00011 in the exponent is now -0.0000018. Thus, following the same procedure, the solution for t is

$$t = \frac{\ln(1/2)}{-0.0000018} \approx 385{,}081 \text{ years.}$$

50. Given the doubling time of 5 hours, we can solve for the bacteria's growth rate;

$$\begin{aligned}2P_0 &= P_0e^{k5}\\ k &= \frac{\ln 2}{5}.\end{aligned}$$

So the growth of the bacteria population is given by:

$$P = P_0e^{\ln(2)t/5}.$$

We want to find t such that

$$3P_0 = P_0e^{\ln(2)t/5}.$$

Therefore we cancel P_0 and apply $\ln$. We get

$$t = \frac{5\ln(3)}{\ln(2)} = 7.925 \text{ hours.}$$

51. (a) The pressure P at 6194 meters is given in terms of the pressure P_0 at sea level to be

$$\begin{aligned}P &= P_0e^{-0.00012h}\\ &= P_0e^{(-0.00012)6194}\\ &= P_0e^{-0.74328}\\ &\approx 0.4756P_0 \quad \text{or about 47.6\% of sea level pressure.}\end{aligned}$$

(b) At $h = 12{,}000$ meters, we have

$$\begin{aligned}P &= P_0e^{-0.00012h}\\ &= P_0e^{(-0.00012)12{,}000}\\ &= P_0e^{-1.44}\\ &\approx 0.2369P_0 \quad \text{or about 23.7\% of sea level pressure.}\end{aligned}$$

52. We know that the y-intercept of the line is at (0,1), so we need one other point to determine the equation of the line. We observe that it intersects the graph of $f(x) = 10^x$ at the point $x = \log 2$. The y-coordinate of this point is then

$$y = 10^x = 10^{\log 2} = 2,$$

so $(\log 2, 2)$ is the point of intersection. We can now find the slope of the line:

$$m = \frac{2-1}{\log 2 - 0} = \frac{1}{\log 2}.$$

Plugging this into the point-slope formula for a line, we have

$$\begin{aligned}y - y_1 &= m(x - x_1)\\ y - 1 &= \frac{1}{\log 2}(x - 0)\\ y &= \frac{1}{\log 2}x + 1 \approx 3.3219x + 1.\end{aligned}$$

53. If t is time in decades, then the number of vehicles, V, in millions, is given by

$$V = 246(1.155)^t.$$

For time t in decades, the number of people, P, in millions, is given by

$$P = 308.7(1.097)^t.$$

There is an average of one vehicle per person when $\frac{V}{P} = 1$, or $V = P$. Thus, we solve for t in the equation:

$$246(1.155)^t = 308.7(1.097)^t,$$

which leads to

$$\left(\frac{1.155}{1.097}\right)^t = \frac{(1.155)^t}{(1.097)^t} = \frac{308.7}{246}$$

Taking logs on both sides, we get

$$t \log \frac{1.155}{1.097} = \log \frac{308.7}{246},$$

so

$$t = \frac{\log(308.7/246)}{\log(1.155/1.097)} = 4.41 \text{ decades.}$$

This model predicts one vehicle per person in 2054

54. We assume exponential decay and solve for k using the half-life:

$$e^{-k(5730)} = 0.5 \quad \text{so} \quad k = 1.21 \cdot 10^{-4}.$$

Now find t, the age of the painting:

$$e^{-1.21 \cdot 10^{-4} t} = 0.995, \quad \text{so} \quad t = \frac{\ln 0.995}{-1.21 \cdot 10^{-4}} = 41.43 \text{ years.}$$

Since Vermeer died in 1675, the painting is a fake.

55. Yes, $\ln(\ln(x))$ means take the ln of the value of the function $\ln x$. On the other hand, $\ln^2(x)$ means take the function $\ln x$ and square it. For example, consider each of these functions evaluated at e. Since $\ln e = 1$, $\ln^2 e = 1^2 = 1$, but $\ln(\ln(e)) = \ln(1) = 0$. See the graphs in Figure 1.37. (Note that $\ln(\ln(x))$ is only defined for $x > 1$.)

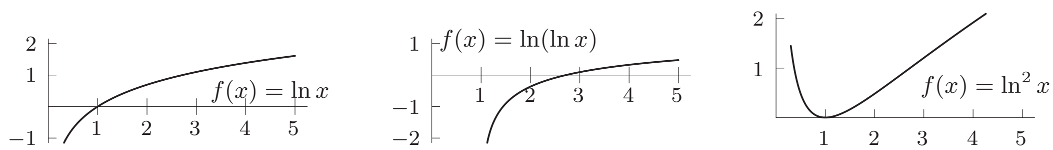


Figure 1.37

56. (a) The y-intercept of $h(x) = \ln(x + a)$ is $h(0) = \ln a$. Thus increasing a increases the y-intercept.
(b) The x-intercept of $h(x) = \ln(x+a)$ is where $h(x) = 0$. Since this occurs where $x+a = 1$, or $x = 1-a$, increasing a moves the x-intercept to the left.

57. The vertical asymptote is where $x + a = 0$, or $x = -a$. Thus increasing a moves the vertical asymptote to the left.

58. (a) The y-intercept of $g(x) = \ln(ax + 2)$ is $g(0) = \ln 2$. Thus increasing a does not effect the y-intercept.
(b) The x-intercept of $g(x) = \ln(ax + 2)$ is where $g(x) = 0$. Since this occurs where $ax + 2 = 1$, or $x = -1/a$, increasing a moves the x-intercept toward the origin. (The intercept is to the left of the origin if $a > 0$ and to the right if $a < 0$.)

59. The vertical asymptote is where $x + 2 = 0$, or $x = -2$, so increasing a does not effect the vertical asymptote.

60. The vertical asymptote is where $ax + 2 = 0$, or $x = -2/a$. Thus increasing a moves the vertical asymptote toward the origin. (The asymptote is to the left of the origin for $a > 0$ and to the right of the origin for $a < 0$.)

Strengthen Your Understanding

61. The function $-\log|x|$ is even, since $|-x| = |x|$, which means $-\log|-x| = -\log|x|$.

62. We have

$$\ln(100x) = \ln(100) + \ln x.$$

In general, $\ln(100x) \neq 100 \cdot \ln x$.

63. One possibility is $f(x) = -x$, because $\ln(-x)$ is only defined if $-x > 0$.

64. One possibility is $f(x) = \ln(x-3)$.

65. True, as seen from the graph.

66. False, since $\log(x-1) = 0$ if $x - 1 = 1$, so $x = 2$.

67. False. The inverse function is $y = 10^x$.

68. False, since $ax + b = 0$ if $x = -b/a$. Thus $y = \ln(ax+b)$ has a vertical asymptote at $x = -b/a$.

Solutions for Section 1.5

Exercises

1. See Figure 1.38.

$$\sin\left(\frac{3\pi}{2}\right) = -1 \quad \text{is negative.}$$
$$\cos\left(\frac{3\pi}{2}\right) = 0$$
$$\tan\left(\frac{3\pi}{2}\right) \quad \text{is undefined.}$$

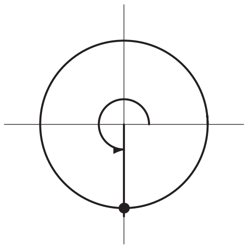


Figure 1.38

2. See Figure 1.39.

$$\sin(2\pi) = 0$$
$$\cos(2\pi) = 1 \text{ is positive.}$$
$$\tan(2\pi) = 0$$

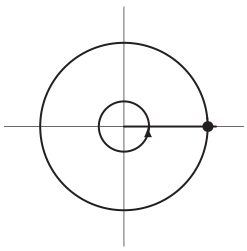


Figure 1.39

3. See Figure 1.40.

$$\begin{aligned} \sin\frac{\pi}{4} &\quad \text{is positive} \\ \cos\frac{\pi}{4} &\quad \text{is positive} \\ \tan\frac{\pi}{4} &\quad \text{is positive} \end{aligned}$$

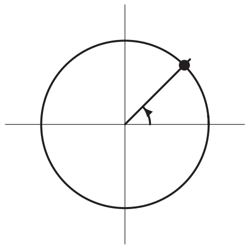


Figure 1.40

4. See Figure 1.41.

$$\begin{aligned} \sin 3\pi &= 0 \\ \cos 3\pi &= -1 \text{ is negative} \\ \tan 3\pi &= 0 \end{aligned}$$

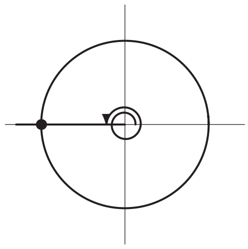


Figure 1.41

5. See Figure 1.42.

$$\begin{aligned} \sin\left(\frac{\pi}{6}\right) &\quad \text{is positive.} \\ \cos\left(\frac{\pi}{6}\right) &\quad \text{is positive.} \\ \tan\left(\frac{\pi}{6}\right) &\quad \text{is positive.} \end{aligned}$$

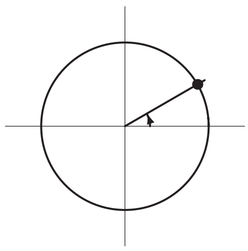


Figure 1.42

6. See Figure 1.43.

$$\sin\frac{4\pi}{3} \text{ is negative}$$
$$\cos\frac{4\pi}{3} \text{ is negative}$$
$$\tan\frac{4\pi}{3} \text{ is positive}$$

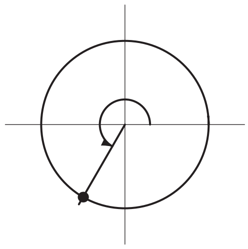


Figure 1.43

7. See Figure 1.44.

$$\sin\left(\frac{-4\pi}{3}\right) \text{ is positive.}$$
$$\cos\left(\frac{-4\pi}{3}\right) \text{ is negative.}$$
$$\tan\left(\frac{-4\pi}{3}\right) \text{ is negative.}$$

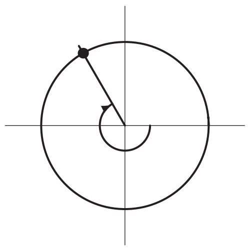


Figure 1.44

8. $4 \text{ radians} \cdot \dfrac{180^\circ}{\pi \text{ radians}} = \left(\dfrac{720}{\pi}\right)^\circ \approx 240^\circ$. See Figure 1.45.

$$\sin 4 \text{ is negative}$$
$$\cos 4 \text{ is negative}$$
$$\tan 4 \text{ is positive.}$$

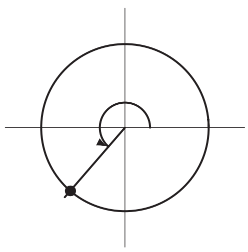


Figure 1.45

9. $-1 \text{ radian} \cdot \frac{180°}{\pi \text{ radians}} = -\left(\frac{180°}{\pi}\right) \approx -60°$. See Figure 1.46.

$$\begin{array}{ll} \sin(-1) & \text{is negative} \\ \cos(-1) & \text{is positive} \\ \tan(-1) & \text{is negative.} \end{array}$$

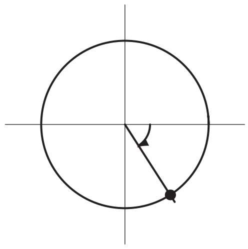


Figure 1.46

10. The period is $2\pi/3$, because when t varies from 0 to $2\pi/3$, the quantity $3t$ varies from 0 to 2π. The amplitude is 7, since the value of the function oscillates between -7 and 7.

11. The period is $2\pi/(1/4) = 8\pi$, because when u varies from 0 to 8π, the quantity $u/4$ varies from 0 to 2π. The amplitude is 3, since the function oscillates between 2 and 8.

12. The period is $2\pi/2 = \pi$, because as x varies from $-\pi/2$ to $\pi/2$, the quantity $2x + \pi$ varies from 0 to 2π. The amplitude is 4, since the function oscillates between 4 and 12.

13. The period is $2\pi/\pi = 2$, since when t increases from 0 to 2, the value of πt increases from 0 to 2π. The amplitude is 0.1, since the function oscillates between 1.9 and 2.1.

14. This graph is a sine curve with period 8π and amplitude 2, so it is given by $f(x) = 2\sin\left(\frac{x}{4}\right)$.

15. This graph is a cosine curve with period 6π and amplitude 5, so it is given by $f(x) = 5\cos\left(\frac{x}{3}\right)$.

16. This graph is an inverted sine curve with amplitude 4 and period π, so it is given by $f(x) = -4\sin(2x)$.

17. This graph is an inverted cosine curve with amplitude 8 and period 20π, so it is given by $f(x) = -8\cos\left(\frac{x}{10}\right)$.

18. This graph has period 6, amplitude 5 and no vertical or horizontal shift, so it is given by

$$f(x) = 5\sin\left(\frac{2\pi}{6}x\right) = 5\sin\left(\frac{\pi}{3}x\right).$$

19. The graph is a cosine curve with period $2\pi/5$ and amplitude 2, so it is given by $f(x) = 2\cos(5x)$.

20. The graph is an inverted sine curve with amplitude 1 and period 2π, shifted up by 2, so it is given by $f(x) = 2 - \sin x$.

21. This can be represented by a sine function of amplitude 3 and period 18. Thus,

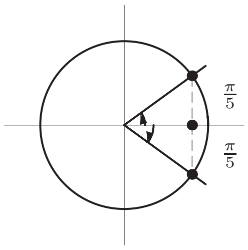
$$f(x) = 3\sin\left(\frac{\pi}{9}x\right).$$

22. This graph is the same as in Problem 14 but shifted up by 2, so it is given by $f(x) = 2\sin\left(\frac{x}{4}\right) + 2$.

23. This graph has period 8, amplitude 3, and a vertical shift of 3 with no horizontal shift. It is given by

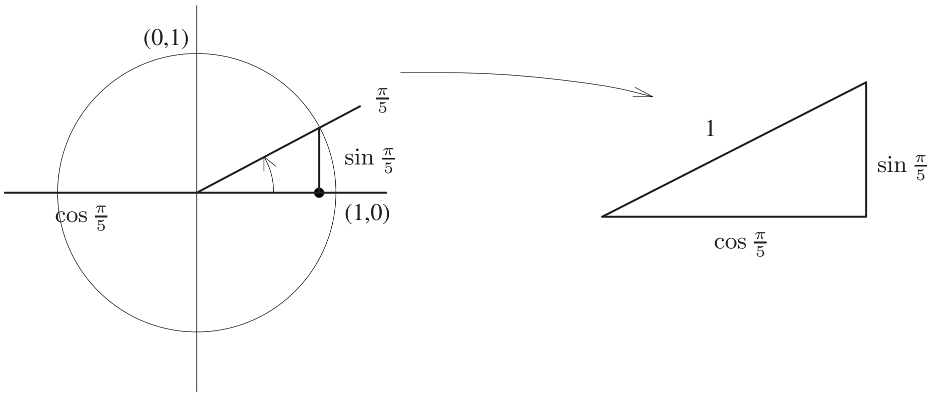
$$f(x) = 3 + 3\sin\left(\frac{2\pi}{8}x\right) = 3 + 3\sin\left(\frac{\pi}{4}x\right).$$

24.



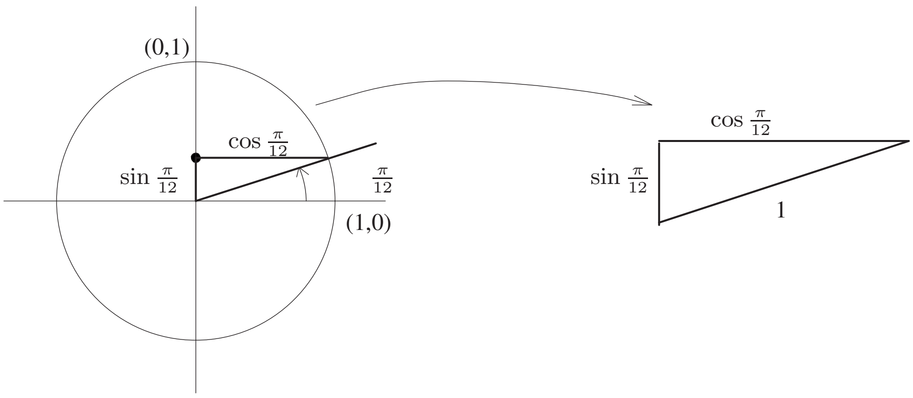
$$\cos\left(-\frac{\pi}{5}\right) = \cos\frac{\pi}{5} \quad \text{(by picture)}$$
$$= 0.809.$$

25.



By the Pythagorean Theorem, $(\cos\frac{\pi}{5})^2 + (\sin\frac{\pi}{5})^2 = 1^2$;
so $(\sin\frac{\pi}{5})^2 = 1 - (\cos\frac{\pi}{5})^2$, and $\sin\frac{\pi}{5} = \sqrt{1-(\cos\frac{\pi}{5})^2} = \sqrt{1-(0.809)^2} \approx 0.588$.
We take the positive square root since by the picture we know that $\sin\frac{\pi}{5}$ is positive.

26.



By the Pythagorean Theorem, $(\cos\frac{\pi}{12})^2+(\sin\frac{\pi}{12})^2 = 1^2$; so $(\cos\frac{\pi}{12})^2 = 1-(\sin\frac{\pi}{12})^2$ and $\cos\frac{\pi}{12} = \sqrt{1-(\sin\frac{\pi}{12})^2} = \sqrt{1-(0.259)^2} \approx 0.966$. We take the positive square root since by the picture we know that $\cos\frac{\pi}{12}$ is positive.

27. We first divide by 5 and then use inverse sine:

$$\frac{2}{5} = \sin(3x)$$
$$\sin^{-1}(2/5) = 3x$$
$$x = \frac{\sin^{-1}(2/5)}{3} \approx 0.1372.$$

There are infinitely many other possible solutions since the sine is periodic.

28. We first isolate $\cos(2x+1)$ and then use inverse cosine:

$$\begin{aligned}1 &= 8\cos(2x+1)-3\\ 4 &= 8\cos(2x+1)\\ 0.5 &= \cos(2x+1)\\ \cos^{-1}(0.5) &= 2x+1\\ x &= \frac{\cos^{-1}(0.5)-1}{2} \approx 0.0236.\end{aligned}$$

There are infinitely many other possible solutions since the cosine is periodic.

29. We first isolate $\tan(5x)$ and then use inverse tangent:

$$\begin{aligned}8 &= 4\tan(5x)\\ 2 &= \tan(5x)\\ \tan^{-1}2 &= 5x\\ x &= \frac{\tan^{-1}2}{5} = 0.221.\end{aligned}$$

There are infinitely many other possible solutions since the tangent is periodic.

30. We first isolate $(2x+1)$ and then use inverse tangent:

$$\begin{aligned}1 &= 8\tan(2x+1)-3\\ 4 &= 8\tan(2x+1)\\ 0.5 &= \tan(2x+1)\\ \arctan(0.5) &= 2x+1\\ x &= \frac{\arctan(0.5)-1}{2} = -0.268.\end{aligned}$$

There are infinitely many other possible solutions since the tangent is periodic.

31. We first isolate $\sin(5x)$ and then use inverse sine:

$$\begin{aligned}8 &= 4\sin(5x)\\ 2 &= \sin(5x).\end{aligned}$$

But this equation has no solution since $-1 \leq \sin(5x) \leq 1$.

Problems

32. (a) $h(t) = 2\cos(t-\pi/2)$
(b) $f(t) = 2\cos t$
(c) $g(t) = 2\cos(t+\pi/2)$

33. $\sin x^2$ is by convention $\sin(x^2)$, which means you square the x first and then take the sine.
$\sin^2 x = (\sin x)^2$ means find $\sin x$ and then square it.
$\sin(\sin x)$ means find $\sin x$ and then take the sine of that.
Expressing each as a composition: If $f(x) = \sin x$ and $g(x) = x^2$, then
$\sin x^2 = f(g(x))$
$\sin^2 x = g(f(x))$
$\sin(\sin x) = f(f(x))$.

34. Suppose P is at the point $(3\pi/2, -1)$ and Q is at the point $(5\pi/2, 1)$. Then

$$\text{Slope } = \frac{1-(-1)}{5\pi/2 - 3\pi/2} = \frac{2}{\pi}.$$

If P had been picked to the right of Q, the slope would have been $-2/\pi$.

35. (a) See Figure 1.47.

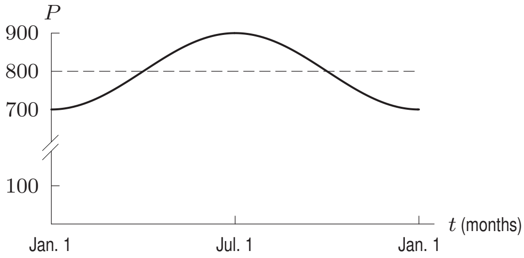


Figure 1.47

(b) Average value of population $= \frac{700+900}{2} = 800$, amplitude $= \frac{900-700}{2} = 100$, and period $= 12$ months, so $B = 2\pi/12 = \pi/6$. Since the population is at its minimum when $t = 0$, we use a negative cosine:

$$P = 800 - 100\cos\left(\frac{\pi t}{6}\right).$$

36. We use a cosine of the form

$$H = A\cos(Bt) + C$$

and choose B so that the period is 24 hours, so $2\pi/B = 24$ giving $B = \pi/12$.

The temperature oscillates around an average value of 60° F, so $C = 60$. The amplitude of the oscillation is 20° F. To arrange that the temperature be at its lowest when $t = 0$, we take A negative, so $A = -20$. Thus

$$A = 60 - 20\cos\left(\frac{\pi}{12}t\right).$$

37. (a) $f(t) = -0.5 + \sin t, \quad g(t) = 1.5 + \sin t, \quad h(t) = -1.5 + \sin t, \quad k(t) = 0.5 + \sin t.$

(b) The values of $g(t)$ are one more than the values of $k(t)$, so $g(t) = 1 + k(t)$. This happens because $g(t) = 1.5 + \sin t = 1 + 0.5 + \sin t = 1 + k(t)$.

(c) Since $-1 \le \sin t \le 1$, adding 1.5 everywhere we get $0.5 \le 1.5 + \sin t \le 2.5$ and since $1.5 + \sin t = g(t)$, we get $0.5 \le g(t) \le 2.5$. Similarly, $-2.5 \le -1.5 + \sin t = h(t) \le -0.5$.

38. Depth $= 7 + 1.5\sin\left(\frac{\pi}{3}t\right)$

39. (a) Beginning at time $t = 0$, the voltage will have oscillated through a complete cycle when $\cos(120\pi t) = \cos(2\pi)$, hence when $t = \frac{1}{60}$ second. The period is $\frac{1}{60}$ second.

(b) V_0 represents the amplitude of the oscillation.

(c) See Figure 1.48.

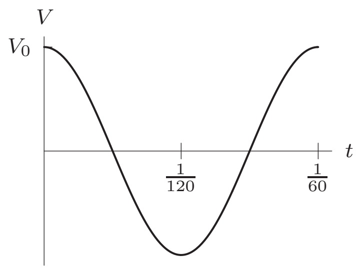


Figure 1.48

40. (a) When the time is t hours after 6 am, the solar panel outputs $f(t) = P(\theta(t))$ watts. So,

$$f(t) = 10\sin\left(\frac{\pi}{14}t\right)$$

where $0 \le t \le 14$ is the number of hours after 6 am.

(b) The graph of $f(t)$ is in Figure 1.49:

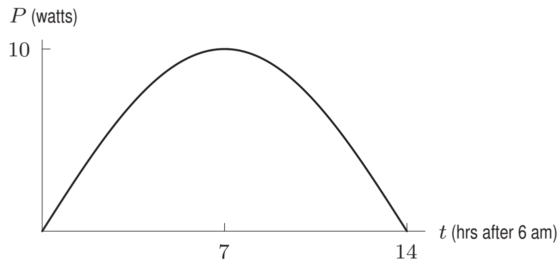


Figure 1.49

(c) The power output is greatest when $\sin(\pi t/14) = 1$. Since $0 \leq \pi t/14 \leq \pi$, the only point in the domain of f at which $\sin(\pi t/14) = 1$ is when $\pi t/14 = \pi/2$. Therefore, the power output is greatest when $t = 7$, that is, at 1 pm. The output at this time will be $f(7) = 10$ watts.

(d) On a typical winter day, there are 9 hours of sun instead of the 14 hours of sun. So, if t is the number of hours since 8 am, the angle between a solar panel and the sun is

$$\phi = \frac{14}{9}\theta = \frac{\pi}{9}t \quad \text{where } 0 \leq t \leq 9.$$

The solar panel outputs $g(t) = P(\phi(t))$ watts:

$$g(t) = 10\sin\left(\frac{\pi}{9}t\right)$$

where $0 \leq t \leq 9$ is the number of hours after 8 am.

41. The function R has period of π, so its graph is as shown in Figure 1.50. The maximum value of the range is v_0^2/g and occurs when $\theta = \pi/4$.

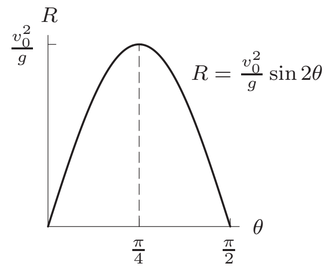


Figure 1.50

42. Over the one-year period, the average value is about 75° and the amplitude of the variation is about $\frac{90-60}{2} = 15^\circ$. The function assumes its minimum value right at the beginning of the year, so we want a negative cosine function. Thus, for t in years, we have the function

$$f(t) = 75 - 15\cos\left(\frac{2\pi}{12}t\right).$$

(Many other answers are possible, depending on how you read the chart.)

43. (a) $D =$ the average depth of the water.
(b) $A =$ the amplitude $= 15/2 = 7.5$.
(c) Period = 12.4 hours. Thus $(B)(12.4) = 2\pi$ so $B = 2\pi/12.4 \approx 0.507$.
(d) C is the time of a high tide.

44. Using the fact that 1 revolution $= 2\pi$ radians and 1 minute = 60 seconds, we have

$$\begin{aligned} 200\frac{\text{rev}}{\text{min}} &= (200)\cdot 2\pi\frac{\text{rad}}{\text{min}} = 200\cdot 2\pi\frac{1}{60}\frac{\text{rad}}{\text{sec}} \\ &\approx \frac{(200)(6.283)}{60} \\ &\approx 20.94 \text{ radians per second.} \end{aligned}$$

Similarly, 500 rpm is equivalent to 52.36 radians per second.

45. 200 revolutions per minute is $\frac{1}{200}$ minutes per revolution, so the period is $\frac{1}{200}$ minutes, or 0.3 seconds.

46. The earth makes one revolution around the sun in one year, so its period is one year.

47. The moon makes one revolution around the earth in about 27.3 days, so its period is 27.3 days $\approx$ one month.

48. (a) The period of the tides is $2\pi/0.5 = 4\pi = 12.566$ hours.

(b) The boat is afloat provided the water is deeper than 2.5 meters, so we need

$$d(t) = 5 + 4.6\sin(0.5t) > 2.5.$$

Figure 1.51 is a graph of $d(t)$, with time t in hours since midnight, $0 \le t \le 24$. The boat leaves at $t = 12$ (midday). To find the latest time the boat can return, we need to solve the equation $d(t) = 5 + 4.6\sin(0.5t) = 2.5$.

A quick way to estimate the solution is to trace along the line $y = 2.5$ in Figure 1.51 until we get to the first point of intersection to the right of $t = 12$. The value we want is about $t = 20$. Thus the water remains deep enough until about 8 pm.

To find t analytically, we solve

$$\begin{aligned} 5 + 4.6\sin(0.5t) &= 2.5 \\ \sin(0.5t) &= -\frac{2.5}{4.6} = -0.5435 \\ t &= \frac{1}{0.5}\arcsin(-0.5435) = -1.149. \end{aligned}$$

This is the value of t immediately to the left of the vertical axis. The water is also 2.5 meters deep one period later at $t = -1.149 + 12.566 = 11.417$. This is shortly before the boat leaves, while the water is rising. We want the next time the time the water is this depth.

The water was at its deepest (that is, $d(t)$ was a maximum) when $t = 12.566/4 = 3.142$. From the figure, the time between when the water was 2.5 meters and when it was deepest was $3.142 + 1.149 = 4.291$ hours. Thus, the value of t that we want is

$$t = 11.417 + 2 \cdot 4.291 = 19.999.$$

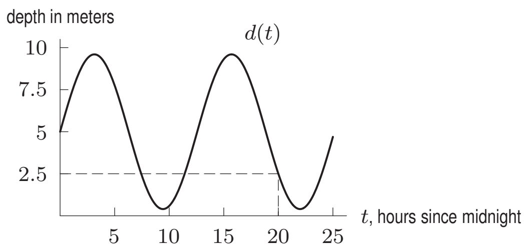


Figure 1.51

49. Since b is a positive constant, f is a vertical shift of $\sin t$ where the midline lies above the t-axis. So f matches Graph C.

Function g is the sum of $\sin t$ plus a linear function $at + b$. We suspect then that the graph of g might periodically oscillate about a line $at + b$, just like the graph of $\sin t$ oscillates about its midline. When adding $at + b$ to $\sin t$, we note that every zero of $\sin t$, $(t, 0)$, gets displaced to a corresponding point $(t, at + b)$ that lies both on the graph of g, and on the line $at + b$. See Figure 1.52. So g matches Graph B.

Function h is the sum of $\sin t$ plus an increasing exponential function $e^{ct} + d$. We suspect then that the graph of g might periodically oscillate about the graph of $e^{ct} + d$, just like the graph of $\sin t$ oscillates about its midline. When adding $e^{ct} + d$ to $\sin t$, we note that every zero of $\sin t$, $(t, 0)$, gets displaced to a corresponding point $(t, e^{ct} + d)$ that lies both on the graph of h, and on the graph of $e^{ct} + d$. See Figure 1.52. So h matches Graph A. Note that the oscillations on the graph of h may not be visible for all t values.

Function r is the sum of $\sin t$ plus an decreasing exponential function $-e^{ct} + b$. We suspect then that the graph of r might periodically oscillate about the graph of $-e^{ct} + b$, just like the graph of $\sin t$ oscillates about its midline. When adding $-e^{ct} + b$ to $\sin t$, we note that every zero of $\sin t$, $(t, 0)$, gets displaced to a corresponding point $(t, -e^{ct} + b)$ that lies both on the graph of r, and on the graph of $-e^{ct} + b$. See Figure 1.52. So r matches Graph D. Note that the oscillations on the graph of r may not be visible for all t values.

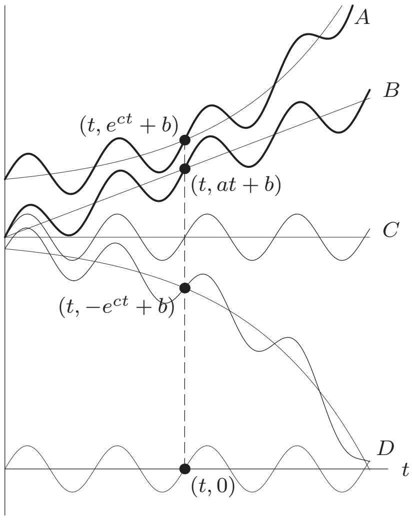


Figure 1.52

50. (a) The monthly mean CO_2 increased about 10 ppm between December 2005 and December 2010. This is because the black curve shows that the December 2005 monthly mean was about 381 ppm, while the December 2010 monthly mean was about 391 ppm. The difference between these two values, $391 - 381 = 10$, gives the overall increase.

(b) The average rate of increase is given by

$$\text{Average monthly increase of monthly mean} = \frac{391-381}{60-0} = \frac{1}{6}\text{ ppm/month.}$$

This tells us that the slope of a linear equation approximating the black curve is $1/6$. Since the vertical intercept is about 381, a possible equation for the approximately linear black curve is

$$y = \frac{1}{6}t + 381,$$

where t is measured in months since December 2005.

(c) The period of the seasonal CO_2 variation is about 12 months since this is approximately the time it takes for the function given by the blue curve to complete a full cycle. The amplitude is about 3.5 since, looking at the blue curve, the average distance between consecutive maximum and minimum values is about 7 ppm. So a possible sinusoidal function for the seasonal CO_2 cycle is

$$y = 3.5\sin\left(\frac{\pi}{6}t\right).$$

(d) Taking $f(t) = 3.5\sin\left(\frac{\pi}{6}t\right)$ and $g(t) = \frac{1}{6}t + 381$, we have

$$h(t) = 3.5\sin\left(\frac{\pi}{6}t\right) + \frac{1}{6}t + 381.$$

See Figure 1.53.

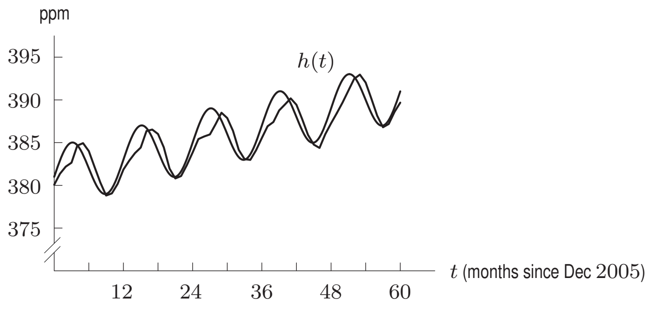


Figure 1.53

51. (a) The period is 2π.

(b) After π, the values of $\cos 2\theta$ repeat, but the values of $2\sin\theta$ do not (in fact, they repeat but flipped over the x-axis). After another π, that is after a total of 2π, the values of $\cos 2\theta$ repeat *again*, and now the values of $2\sin\theta$ repeat also, so the function $2\sin\theta + 3\cos 2\theta$ repeats at that point.

52. Figure 1.54 shows that the cross-sectional area is one rectangle of area hw and two triangles. Each triangle has height h and base x, where

$$\frac{h}{x} = \tan\theta \quad \text{so} \quad x = \frac{h}{\tan\theta}.$$

$$\text{Area of triangle} = \frac{1}{2}xh = \frac{h^2}{2\tan\theta}$$

$$\begin{aligned}\text{Total area} &= \text{Area of rectangle} + 2(\text{Area of triangle})\\ &= hw + 2\cdot\frac{h^2}{2\tan\theta} = hw + \frac{h^2}{\tan\theta}.\end{aligned}$$

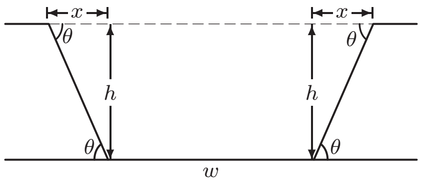


Figure 1.54

53. (a) Two solutions: 0.4 and 2.7. See Figure 1.55.

(b) $\arcsin(0.4)$ is the first solution approximated above; the second is an approximation to $\pi - \arcsin(0.4)$.

(c) By symmetry, there are two solutions: -0.4 and -2.7.

(d) $-0.4 \approx -\arcsin(0.4)$ and $-2.7 \approx -(\pi - \arcsin(0.4)) = \arcsin(0.4) - \pi$.

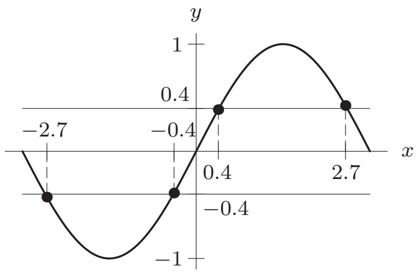


Figure 1.55

54. The ramp in Figure 1.56 rises 1 ft over a horizontal distance of x ft.

(a) For a 1 ft rise over 12 ft, the angle in radians is $\theta = \arctan(1/12) = 0.0831$. To find the angle in degrees, multiply by $180/\pi$. Hence

$$\theta = \frac{180}{\pi}\arctan\frac{1}{12} = 4.76°.$$

(b) We have

$$\theta = \frac{180}{\pi}\arctan\frac{1}{8} = 7.13°.$$

(c) We have

$$\theta = \frac{180}{\pi}\arctan\frac{1}{20} = 2.86°.$$

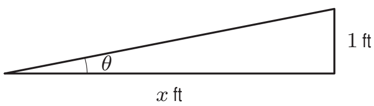


Figure 1.56

55. (a) A table of values for $g(x)$ is:

x	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
$\arccos x$	3.14	2.50	2.21	1.98	1.77	1.57	1.37	1.16	0.93	0.64	0

(b) See Figure 1.57.

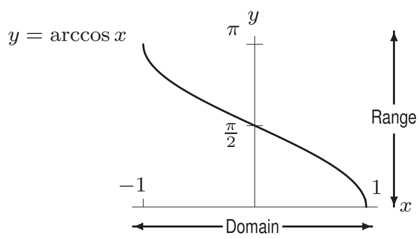


Figure 1.57

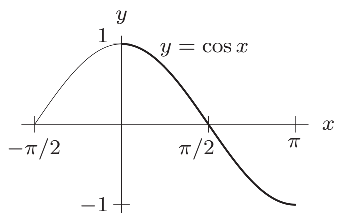


Figure 1.58

(c) The domain of $\arccos$ is $-1 \leq x \leq 1$, because its inverse, cosine, takes all values from -1 to 1. The domains of $\arccos$ and $\arcsin$ are the same because their inverses, sine and cosine, have the same range.

(d) Figure 1.57 shows that the range of $y = \arccos x$ is $0 \leq \theta \leq \pi$.

(e) The range of an inverse function is the domain of the original function. The arcsine is the inverse function to the piece of the sine having domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Hence, the range of the arcsine is $[-\frac{\pi}{2}, \frac{\pi}{2}]$. But the piece of the cosine having domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ does not have an inverse, because there are horizontal lines that intersect its graph twice. Instead, we define arccosine to be the inverse of the piece of cosine having domain $[0, \pi]$, so the range of arccosine is $[0, \pi]$, which is different from the range of arcsine. See Figure 1.58.

Strengthen Your Understanding

56. Increasing the value of B decreases the period. For example, $f(x) = \sin x$ has period 2π, whereas $g(x) = \sin(2x)$ has period π.

57. The maximum value of $A\sin(Bx)$ is A, so the maximum value of $y = A\sin(Bx) + C$ is $y = A + C$.

58. For $B > 0$, the period of $y = \sin(Bx)$ is $2\pi/B$. Thus, we want

$$\frac{2\pi}{B} = 23 \quad \text{so} \quad B = \frac{2\pi}{23}.$$

The function is $f(x) = \sin(2\pi x/23)$

59. The midline is $y = (1200 + 2000)/2 = 1600$ and the amplitude is $y = (2000 - 1200)/2 = 400$, so a possible function is

$$f(x) = 400(\cos x) + 1600.$$

60. False, since $\cos\theta$ is decreasing and $\sin\theta$ is increasing.

61. False. The period is $2\pi/(0.05\pi) = 40$

62. True. The period is $2\pi/(200\pi) = 1/100$ seconds. Thus, the function executes 100 cycles in 1 second.

63. False. If $\theta = \pi/2, 3\pi/2, 5\pi/2 \ldots$, then $\theta - \pi/2 = 0, \pi, 2\pi \ldots$, and the tangent is defined (it is zero) at these values.

64. False: When $x < 0$, we have $\sin|x| = \sin(-x) = -\sin x \neq \sin x$.

65. False: When $\pi < x < 2\pi$, we have $\sin|x| = \sin x < 0$ but $|\sin x| > 0$.

66. False: When $\pi/2 < x < 3\pi/2$, we have $\cos|x| = \cos x < 0$ but $|\cos x| > 0$.

67. True: Since $\cos(-x) = \cos x$, $\cos|x| = \cos x$.

68. False. For example, $\sin(0) \neq \sin((2\pi)^2)$, since $\sin(0) = 0$ but $\sin((2\pi)^2) = 0.98$.

69. True. Since $\sin(\theta + 2\pi) = \sin\theta$ for all θ, we have $g(\theta + 2\pi) = e^{\sin(\theta+2\pi)} = e^{\sin\theta} = g(\theta)$ for all θ.

70. False. A counterexample is given by $f(x) = \sin x$, which has period 2π, and $g(x) = x^2$. The graph of $f(g(x)) = \sin(x^2)$ in Figure 1.59 is not periodic with period 2π.

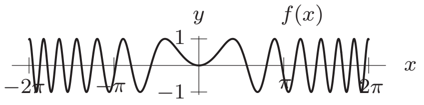


Figure 1.59

71. True. If $g(x)$ has period k, then $g(x + k) = g(x)$. Thus we have

$$f(g(x + k)) = f(g(x))$$

which shows that $f(g(x))$ is periodic with period k.

72. True, since $|\sin(-x)| = |-\sin x| = \sin x$.

Solutions for Section 1.6

Exercises

1. As $x \to \infty$, $y \to \infty$.
As $x \to -\infty$, $y \to -\infty$.

2. As $x \to \infty$, $y \to \infty$.
As $x \to -\infty$, $y \to 0$.

3. Since $f(x)$ is an even power function with a negative leading coefficient, it follows that $f(x) \to -\infty$ as $x \to +\infty$ and $f(x) \to -\infty$ as $x \to -\infty$.

4. Since $f(x)$ is an odd power function with a positive leading coefficient, it follows that $f(x) \to +\infty$ as $x \to +\infty$ and $f(x) \to -\infty$ as $x \to -\infty$.

5. As $x \to \pm\infty$, the lower-degree terms of $f(x)$ become insignificant, and $f(x)$ becomes approximated by the highest degree term of $5x^4$. Thus, as $x \to \pm\infty$, we see that $f(x) \to +\infty$.

6. As $x \to \pm\infty$, the lower-degree terms of $f(x)$ become insignificant, and $f(x)$ becomes approximated by the highest degree term of $-5x^3$. Thus, as $x \to +\infty$, we see that $f(x) \to -\infty$ and as $x \to -\infty$, we see that $f(x) \to +\infty$.

7. As $x \to \pm\infty$, the lower-degree terms of $f(x)$ become insignificant, and $f(x)$ becomes approximated by the highest degree terms in the numerator and denominator. Thus, as $x \to \pm\infty$, we see that $f(x)$ behaves like $\dfrac{3x^2}{x^2} = 3$. We have $f(x) \to 3$ as $x \to \pm\infty$.

8. As $x \to \pm\infty$, the lower-degree terms of $f(x)$ become insignificant, and $f(x)$ becomes approximated by the highest degree terms in the numerator and denominator. Thus, as $x \to \pm\infty$, we see that $f(x)$ behaves like $\dfrac{-3x^3}{2x^3} = -3/2$. We have $f(x) \to -3/2$ as $x \to \pm\infty$.

9. As $x \to \pm\infty$, we see that $3x^{-4}$ gets closer and closer to 0, so $f(x) \to 0$ as $x \to \pm\infty$.

10. As $x \to +\infty$, we have $f(x) \to +\infty$. As $x \to -\infty$, we have $f(x) \to 0$.

11. The power function with the higher power dominates as $x \to \infty$, so $0.2x^5$ is larger.

12. An exponential growth function always dominates a power function as $x \to \infty$, so $10e^{0.1x}$ is larger.

13. An exponential growth function always dominates a power function as $x \to \infty$, so 1.05^x is larger.

14. The lower-power terms in a polynomial become insignificant as $x \to \infty$, so we are comparing $2x^4$ to the leading term $10x^3$. In comparing two power functions, the higher power dominates as $x \to \infty$, so $2x^4$ is larger.

15. The lower-power terms in a polynomial become insignificant as $x \to \infty$, so we are comparing the leading term $20x^4$ to the leading term $3x^5$. In comparing two power functions, the higher power dominates as $x \to \infty$, so the polynomial with leading term $3x^5$ is larger. As $x \to \infty$, we see that $25 - 40x^2 + x^3 + 3x^5$ is larger.

16. A power function with positive exponent dominates a log function, so as $x \to \infty$, we see that $\sqrt{x}$ is larger.

17. (I) **(a)** Minimum degree is 3 because graph turns around twice.
(b) Leading coefficient is negative because $y \to -\infty$ as $x \to \infty$.
(II) **(a)** Minimum degree is 4 because graph turns around three times.
(b) Leading coefficient is positive because $y \to \infty$ as $x \to \infty$.
(III) **(a)** Minimum degree is 4 because graph turns around three times.
(b) Leading coefficient is negative because $y \to -\infty$ as $x \to \infty$.
(IV) **(a)** Minimum degree is 5 because graph turns around four times.
(b) Leading coefficient is negative because $y \to -\infty$ as $x \to \infty$.
(V) **(a)** Minimum degree is 5 because graph turns around four times.
(b) Leading coefficient is positive because $y \to \infty$ as $x \to \infty$.

18. (a) From the x-intercepts, we know the equation has the form

$$y = k(x+2)(x-1)(x-5).$$

Since $y = 2$ when $x = 0$,

$$\begin{aligned} 2 &= k(2)(-1)(-5) = k \cdot 10 \\ k &= \frac{1}{5}. \end{aligned}$$

Thus we have

$$y = \frac{1}{5}(x+2)(x-1)(x-5).$$

19. (a) Because our cubic has a root at 2 and a double root at -2, it has the form

$$y = k(x+2)(x+2)(x-2).$$

Since $y = 4$ when $x = 0$,

$$\begin{aligned} 4 &= k(2)(2)(-2) = -8k, \\ k &= -\frac{1}{2}. \end{aligned}$$

Thus our equation is

$$y = -\frac{1}{2}(x+2)^2(x-2).$$

20. $f(x) = k(x+3)(x-1)(x-4) = k(x^3 - 2x^2 - 11x + 12)$, where $k < 0$. ($k \approx -\frac{1}{6}$ if the horizontal and vertical scales are equal; otherwise one can't tell how large k is.)

21. $f(x) = kx(x+3)(x-4) = k(x^3 - x^2 - 12x)$, where $k < 0$. ($k \approx -\frac{2}{9}$ if the horizontal and vertical scales are equal; otherwise one can't tell how large k is.)

22. $f(x) = k(x+2)(x-1)(x-3)(x-5) = k(x^4 - 7x^3 + 5x^2 + 31x - 30)$, where $k > 0$. ($k \approx \frac{1}{15}$ if the horizontal and vertical scales are equal; otherwise one can't tell how large k is.)

23. $f(x) = k(x+2)(x-2)^2(x-5) = k(x^4 - 7x^3 + 6x^2 + 28x - 40)$, where $k < 0$. ($k \approx -\frac{1}{15}$ if the scales are equal; otherwise one can't tell how large k is.)

24. There are only two functions, h and p, which can be put in the form $y = Cb^x$, where C and b are constants:

$$p(x) = \frac{a^3 b^x}{c} = (a^3/c)b^x, \quad \text{where } C = a^3/c \text{ since } a, c \text{ are constants.}$$

$$h(x) = \frac{-1}{5^{x-2}} = -5^{-(x-2)} = -5^{-x+2} = (-25)5^{-x}.$$

Thus, h and p are the only exponential functions.

25. There is only one function, r, who can be put in the form $y = Ax^2 + Bx + C$:

$$r(x) = -x + b - \sqrt{cx^4} = -\sqrt{c}\,x^2 - x + b, \quad \text{where } A = -\sqrt{c}, \text{ since } c \text{ is a constant.}$$

Thus, r is the only quadratic function.

26. There is only one function, q, which is linear. Function q is a constant linear function whose vertical intercept is the constant ab^2/c, since a, b and c are constants.

Problems

27. Consider the end behavior of the graph; that is, as $x \to +\infty$ and $x \to -\infty$. The ends of a degree 5 polynomial are in Quadrants I and III if the leading coefficient is positive or in Quadrants II and IV if the leading coefficient is negative. Thus, there must be at least one root. Since the degree is 5, there can be no more than 5 roots. Thus, there may be 1, 2, 3, 4, or 5 roots. Graphs showing these five possibilities are shown in Figure 1.60.

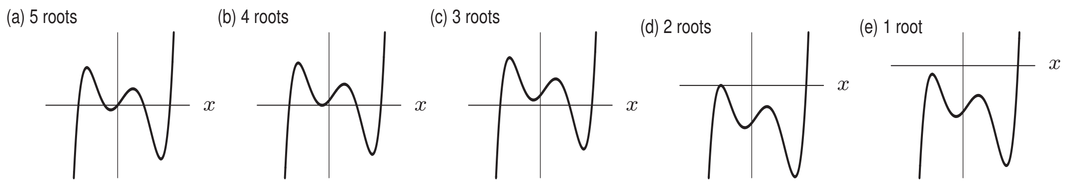


Figure 1.60

28. $g(x) = 2x^2$, $h(x) = x^2 + k$ for any $k > 0$. Notice that the graph is symmetric about the y-axis and $\lim_{x\to\infty} f(x) = 2$.

29. The graphs of both these functions will resemble that of x^3 on a large enough window. One way to tackle the problem is to graph them both (along with x^3 if you like) in successively larger windows until the graphs come together. In Figure 1.61, f, g and x^3 are graphed in four windows. In the largest of the four windows the graphs are indistinguishable, as required. Answers may vary.

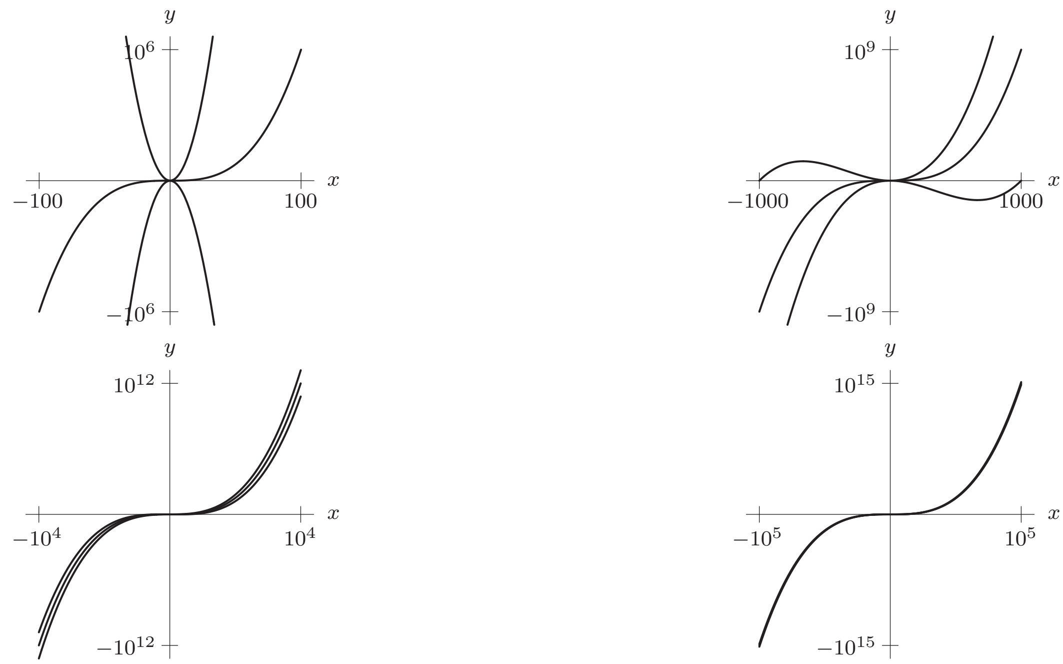


Figure 1.61

30. (a) A polynomial has the same end behavior as its leading term, so this polynomial behaves as $-5x^4$ globally. Thus we have:
$f(x) \to -\infty$ as $x \to -\infty$, and $f(x) \to -\infty$ as $x \to +\infty$.
(b) Polynomials behave globally as their leading term, so this rational function behaves globally as $(3x^2)/(2x^2)$, or $3/2$. Thus we have:
$f(x) \to 3/2$ as $x \to -\infty$, and $f(x) \to 3/2$ as $x \to +\infty$.
(c) We see from a graph of $y = e^x$ that
$f(x) \to 0$ as $x \to -\infty$, and $f(x) \to +\infty$ as $x \to +\infty$.

31. Substituting $w = 65$ and $h = 160$, we have
(a)
$$s = 0.01(65^{0.25})(160^{0.75}) = 1.3 \text{ m}^2.$$
(b) We substitute $s = 1.5$ and $h = 180$ and solve for w:
$$1.5 = 0.01w^{0.25}(180^{0.75}).$$
We have
$$w^{0.25} = \frac{1.5}{0.01(180^{0.75})} = 3.05.$$
Since $w^{0.25} = w^{1/4}$, we take the fourth power of both sides, giving
$$w = 86.8 \text{ kg}.$$
(c) We substitute $w = 70$ and solve for h in terms of s:
$$s = 0.01(70^{0.25})h^{0.75},$$
so
$$h^{0.75} = \frac{s}{0.01(70^{0.25})}.$$
Since $h^{0.75} = h^{3/4}$, we take the $4/3$ power of each side, giving
$$h = \left(\frac{s}{0.01(70^{0.25})}\right)^{4/3} = \frac{s^{4/3}}{(0.01^{4/3})(70^{1/3})}$$
so
$$h = 112.6s^{4/3}.$$

32. Let $D(v)$ be the stopping distance required by an Alpha Romeo as a function of its velocity. The assumption that stopping distance is proportional to the square of velocity is equivalent to the equation
$$D(v) = kv^2$$
where k is a constant of proportionality. To determine the value of k, we use the fact that $D(70) = 177$.
$$D(70) = k(70)^2 = 177.$$
Thus,
$$k = \frac{177}{70^2} \approx 0.0361.$$
It follows that
$$D(35) = \left(\frac{177}{70^2}\right)(35)^2 = \frac{177}{4} = 44.25 \text{ ft}$$
and
$$D(140) = \left(\frac{177}{70^2}\right)(140)^2 = 708 \text{ ft}.$$
Thus, at half the speed it requires one fourth the distance, whereas at twice the speed it requires four times the distance, as we would expect from the equation. (We could in fact have figured it out that way, without solving for k explicitly.)

33. (a) Since the rate R varies directly with the fourth power of the radius r, we have the formula
$$R = kr^4$$
where k is a constant.

(b) Given $R = 400$ for $r = 3$, we can determine the constant k.

$$400 = k(3)^4$$
$$400 = k(81)$$
$$k = \frac{400}{81} \approx 4.938.$$

So the formula is

$$R = 4.938r^4$$

(c) Evaluating the formula above at $r = 5$ yields

$$R = 4.928(5)^4 = 3086.42\frac{\text{cm}^3}{\text{sec}}.$$

34. Let us represent the height by h. Since the volume is V, we have

$$x^2h = V.$$

Solving for h gives

$$h = \frac{V}{x^2}.$$

The graph is in Figure 1.62. We are assuming V is a positive constant.

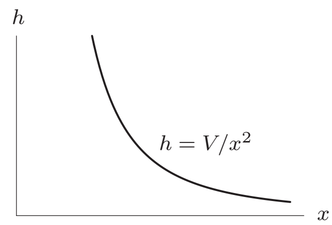


Figure 1.62

35. (a) Let the height of the can be h. Then

$$V = \pi r^2 h.$$

The surface area consists of the area of the ends (each is πr^2) and the curved sides (area $2\pi rh$), so

$$S = 2\pi r^2 + 2\pi rh.$$

Solving for h from the formula for V, we have

$$h = \frac{V}{\pi r^2}.$$

Substituting into the formula for S, we get

$$S = 2\pi r^2 + 2\pi r \cdot \frac{V}{\pi r^2} = 2\pi r^2 + \frac{2V}{r}.$$

(b) For large r, the $2V/r$ term becomes negligible, meaning $S \approx 2\pi r^2$, and thus $S \to \infty$ as $r \to \infty$.
(c) The graph is in Figure 1.63.

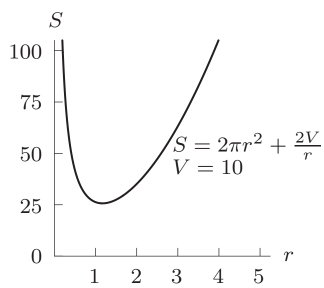


Figure 1.63

36. To find the horizontal asymptote, we look at end behavior. As $x \to \pm\infty$, the lower-degree terms of $f(x)$ become insignificant, and $f(x)$ becomes approximated by the highest degree term of the numerator and denominator. Thus, as $x \to \pm\infty$, we see that
$$f(x) \to \frac{5x}{2x} = \frac{5}{2}.$$
There is a horizontal asymptote at $y = 5/2$.
To find the vertical asymptotes, we set the denominator equal to zero. When $2x + 3 = 0$, we have $x = -3/2$ so there is a vertical asymptote at $x = -3/2$.

37. To find the horizontal asymptote, we look at end behavior. As $x \to \pm\infty$, the lower-degree terms of $f(x)$ become insignificant, and $f(x)$ becomes approximated by the highest degree term of the numerator and denominator. Thus, as $x \to \pm\infty$, we see that
$$f(x) \to \frac{x^2}{x^2} = 1.$$
There is a horizontal asymptote at $y = 1$.
To find the vertical asymptotes, we set the denominator equal to zero. When $x^2 - 4 = 0$, we have $x = \pm 2$ so there are vertical asymptotes at $x = -2$ and at $x = 2$.

38. To find the horizontal asymptote, we look at end behavior. As $x \to \pm\infty$, the lower-degree terms of $f(x)$ become insignificant, and $f(x)$ becomes approximated by the highest degree term of the numerator and denominator. Thus, as $x \to \pm\infty$, we see that
$$f(x) \to \frac{5x^3}{x^3} = 5.$$
There is a horizontal asymptote at $y = 5$.
To find the vertical asymptotes, we set the denominator equal to zero. When $x^3 - 27 = 0$, we have $x = 3$ so there is a vertical asymptote at $x = 3$.

39. (a) The object starts at $t = 0$, when $s = v_0(0) - g(0)^2/2 = 0$. Thus it starts on the ground, with zero height.

(b) The object hits the ground when $s = 0$. This is satisfied at $t = 0$, before it has left the ground, and at some later time t that we must solve for.
$$0 = v_0 t - gt^2/2 = t\,(v_0 - gt/2)$$
Thus $s = 0$ when $t = 0$ and when $v_0 - gt/2 = 0$, i.e., when $t = 2v_0/g$. The starting time is $t = 0$, so it must hit the ground at time $t = 2v_0/g$.

(c) The object reaches its maximum height halfway between when it is released and when it hits the ground, or at
$$t = (2v_0/g)/2 = v_0/g.$$

(d) Since we know the time at which the object reaches its maximum height, to find the height it actually reaches we just use the given formula, which tells us s at any given t. Substituting $t = v_0/g$,
$$\begin{aligned} s &= v_0\left(\frac{v_0}{g}\right) - \frac{1}{2}g\left(\frac{v_0^2}{g^2}\right) = \frac{v_0^2}{g} - \frac{v_0^2}{2g} \\ &= \frac{2v_0^2 - v_0^2}{2g} = \frac{v_0^2}{2g}. \end{aligned}$$

40. The pomegranate is at ground level when $f(t) = -16t^2 + 64t = -16t(t-4) = 0$, so when $t = 0$ or $t = 4$. At time $t = 0$ it is thrown, so it must hit the ground at $t = 4$ seconds. The symmetry of its path with respect to time may convince you that it reaches its maximum height after 2 seconds. Alternatively, we can think of the graph of $f(t) = -16t^2 + 64t = -16(t-2)^2 + 64$, which is a downward parabola with vertex (i.e., highest point) at $(2, 64)$. The maximum height is $f(2) = 64$ feet.

41. (a) (i) If $(1, 1)$ is on the graph, we know that
$$1 = a(1)^2 + b(1) + c = a + b + c.$$
(ii) If $(1, 1)$ is the vertex, then the axis of symmetry is $x = 1$, so
$$-\frac{b}{2a} = 1,$$
and thus
$$a = -\frac{b}{2}, \text{ so } b = -2a.$$
But to be the vertex, $(1, 1)$ must also be on the graph, so we know that $a + b + c = 1$. Substituting $b = -2a$, we get $-a + c = 1$, which we can rewrite as $a = c - 1$, or $c = 1 + a$.

(iii) For $(0, 6)$ to be on the graph, we must have $f(0) = 6$. But $f(0) = a(0^2) + b(0) + c = c$, so $c = 6$.

(b) To satisfy all the conditions, we must first, from (a)(iii), have $c = 6$. From (a)(ii), $a = c - 1$ so $a = 5$. Also from (a)(ii), $b = -2a$, so $b = -10$. Thus the completed equation is

$$y = f(x) = 5x^2 - 10x + 6,$$

which satisfies all the given conditions.

42. The function is a cubic polynomial with positive leading coefficient. Since the figure given in the text shows that the function turns around once, we know that the function has the shape shown in Figure 1.64. The function is below the x-axis for $x = 5$ in the given graph, and we know that it goes to $+\infty$ as $x \to +\infty$ because the leading coefficient is positive. Therefore, there are exactly three zeros. Two zeros are shown, and occur at approximately $x = -1$ and $x = 3$. The third zero must be to the right of $x = 10$ and so occurs for some $x > 10$.

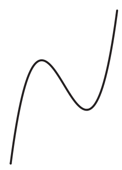


Figure 1.64

43. We use the fact that at a constant speed, Time = Distance/Speed. Thus,

$$\begin{aligned}\text{Total time} &= \text{Time running} + \text{Time walking} \\ &= \frac{3}{x} + \frac{6}{x-2}.\end{aligned}$$

Horizontal asymptote: x-axis.
Vertical asymptote: $x = 0$ and $x = 2$.

44. (a) II and III because in both cases, the numerator and denominator each have x^2 as the highest power, with coefficient $= 1$. Therefore,

$$y \to \frac{x^2}{x^2} = 1 \quad \text{as } x \to \pm\infty.$$

(b) I, since

$$y \to \frac{x}{x^2} = 0 \quad \text{as } x \to \pm\infty.$$

(c) II and III, since replacing x by $-x$ leaves the graph of the function unchanged.
(d) None
(e) III, since the denominator is zero and $f(x)$ tends to $\pm\infty$ when $x = \pm 1$.

45. $h(t)$ cannot be of the form ct^2 or kt^3 since $h(0.0) = 2.04$. Therefore $h(t)$ must be the exponential, and we see that the ratio of successive values of h is approximately 1.5. Therefore $h(t) = 2.04(1.5)^t$. If $g(t) = ct^2$, then $c = 3$ since $g(1.0) = 3.00$. However, $g(2.0) = 24.00 \neq 3 \cdot 2^2$. Therefore $g(t) = kt^3$, and using $g(1.0) = 3.00$, we obtain $g(t) = 3t^3$. Thus $f(t) = ct^2$, and since $f(2.0) = 4.40$, we have $f(t) = 1.1t^2$.

46. The graphs are shown in Figure 1.65.

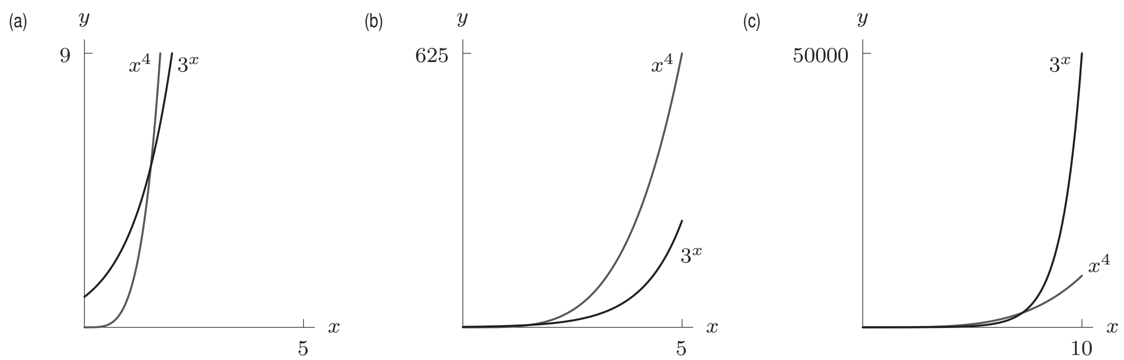


Figure 1.65

47. (a) $R(P) = kP(L-P)$, where k is a positive constant.
(b) A possible graph is in Figure 1.66.

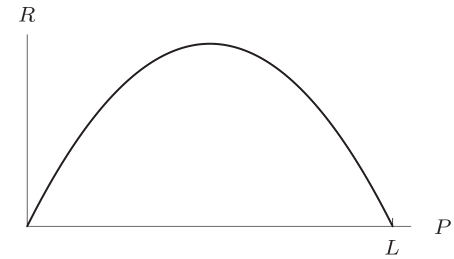


Figure 1.66

48. Since the parabola opens upward, we must have $a > 0$. To determine a relationship between x and y at the point of intersection P, we eliminate a from the parabola and circle equations. Since $y = x^2/a$, we have $a = x^2/y$. Putting this into the circle equation gives $x^2 + y^2 = 2x^4/y^2$. Rewrite this as

$$\begin{aligned} x^2y^2 + y^4 &= 2x^4 \\ y^4 + x^2y^2 - 2x^4 &= 0 \\ (y^2 + 2x^2)(y^2 - x^2) &= 0. \end{aligned}$$

This means $x^2 = y^2$ (since y^2 cannot equal $-2x^2$). Thus $x = y$ since P is in the first quadrant. So P moves out along the line $y = x$ through the origin.

49. (a) The graph is shown in Figure 1.67. The graph represented by the exact formula has a vertical asymptote where the denominator is undefined. This happens when

$$1 - \frac{v^2}{c^2} = 0, \quad \text{or at } v^2 = c^2.$$

Since $v > 0$, the graph of the exact formula has a vertical asymptote where

$$v = c = 3 \cdot 10^8 \text{m/sec}.$$

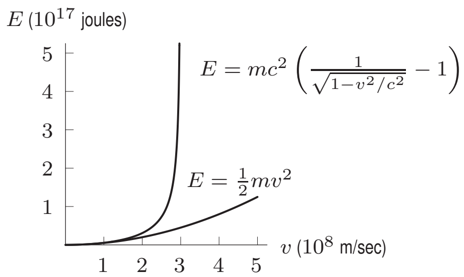


Figure 1.67

(b) The first formula does not give a good approximation to the exact formula when the graphs are not close together. This happens for $v > 1.5 \cdot 10^8$ m/sec. For $v < 1.5 \cdot 10^8$ m/sec, the graphs look close together. However, the vertical scale we are using is so large and the graphs are so close to the v-axis that a more careful analysis should be made. We should zoom in and redraw the graph.

Strengthen Your Understanding

50. The graph of a polynomial of degree 5 cuts the horizontal axis at most five times, but it could be fewer. For example, $f(x) = x^5$ cuts the x-axis only once.

51. The rational function $f(x) = (x^3 + 1)/x$ has no horizontal asymptotes. To see this, observe that

$$y = f(x) = \frac{x^3 + 1}{x} \approx \frac{x^3}{x} = x^2$$

for large x. Thus, $y \to \infty$ as $x \to \pm\infty$.

52. One possibility is $p(x) = (x - 1)(x - 2)(x - 3) = x^3 - 6x^2 + 11x - 6$.

53. A possible function is

$$f(x) = \frac{3x}{x - 10}.$$

54. One possibility is

$$f(x) = \frac{1}{x^2 + 1}.$$

55. Let $f(x) = \dfrac{1}{x + 7\pi}$. Other answers are possible.

56. Let $f(x) = \dfrac{1}{(x-1)(x-2)(x-3)\cdots(x-16)(x-17)}$. This function has an asymptote corresponding to every factor in the denominator. Other answers are possible.

57. The function $f(x) = \dfrac{x - 1}{x - 2}$ has $y = 1$ as the horizontal asymptote and $x = 2$ as the vertical asymptote. These lines cross at the point $(2, 1)$. Other answers are possible.

58. False. The polynomial $f(x) = x^2 + 1$, with degree 2, has no real zeros.

59. True. If the degree of the polynomial, $p(x)$, is n, then the leading term is $a_n x^n$ with $a_n \neq 0$.

If n is odd and a_n is positive, $p(x)$ tends toward ∞ as $x \to \infty$ and $p(x)$ tends toward $-\infty$ as $x \to -\infty$. Since the graph of $p(x)$ has no breaks in it, the graph must cross the x-axis at least once.

If n is odd and a_n is negative, a similar argument applies, with the signs reversed, but leading to the same conclusion.

60. (a), (c), (d), (e), (b). Notice that $f(x)$ and $h(x)$ are decreasing functions, with $f(x)$ being negative. Power functions grow slower than exponential growth functions, so $k(x)$ is next. Now order the remaining exponential functions, where functions with larger bases grow faster.

Solutions for Section 1.7

Exercises

1. Yes, because $x - 2$ is not zero on this interval.

2. No, because $x - 2 = 0$ at $x = 2$.

3. Yes, because $2x - 5$ is positive for $3 \le x \le 4$.

4. Yes, because the denominator is never zero.

5. Yes, because $2x + x^{2/3}$ is defined for all x.

6. No, because $2x + x^{-1}$ is undefined at $x = 0$.

7. No, because $\cos(\pi/2) = 0$.

8. No, because $\sin 0 = 0$.

9. No, because $e^x - 1 = 0$ at $x = 0$.

10. Yes, because $\cos\theta$ is not zero on this interval.

11. We have that $f(0) = -1 < 0$ and $f(1) = 1 > 0$ and that f is continuous. Thus, by the Intermediate Value Theorem applied to $k = 0$, there is a number c in $[0, 1]$ such that $f(c) = k = 0$.

12. We have that $f(0) = 1 > 0$ and $f(1) = e - 3 < 0$ and that f is continuous. Thus, by the Intermediate Value Theorem applied to $k = 0$, there is a number c in $[0, 1]$ such that $f(c) = k = 0$.

13. We have that $f(0) = -1 < 0$ and $f(1) = 1 - \cos 1 > 0$ and that f is continuous. Thus, by the Intermediate Value Theorem applied to $k = 0$, there is a number c in $[0, 1]$ such that $f(c) = k = 0$.

14. Since f is not continuous at $x = 0$, we consider instead the smaller interval $[0.01, 1]$. We have that $f(0.01) = 2^{0.01} - 100 < 0$ and $f(1) = 2 - 1/1 = 1 > 0$ and that f is continuous. Thus, by the Intermediate Value Theorem applied to $k = 0$, there is a number c in $[0.01, 1]$, and hence in $[0, 1]$, such that $f(c) = k = 0$.

15. (a) At $x = 1$, on the line $y = x$, we have $y = 1$. At $x = 1$, on the parabola $y = x^2$, we have $y = 1$. Thus, $f(x)$ is continuous. See Figure 1.68.

(b) At $x = 3$, on the line $y = x$, we have $y = 3$. At $x = 3$, on the parabola $y = x^2$, we have $y = 9$. Thus, $g(x)$ is not continuous. See Figure 1.69.

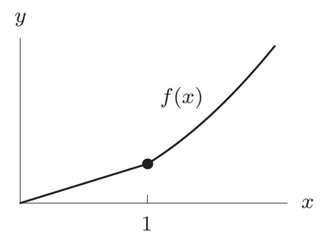


Figure 1.68

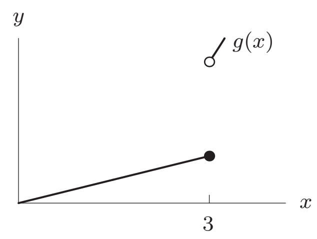


Figure 1.69

Problems

16. (a) Even if the car stops to refuel, the amount of fuel in the tank changes smoothly, so the fuel in the tank is a continuous function; the quantity of fuel cannot suddenly change from one value to another.

(b) Whenever a student joins or leaves the class the number jumps up or down immediately by 1 so this is not a continuous function, unless the enrollment does not change at all.

(c) Whenever the oldest person dies the value of the function jumps down to the age of the next oldest person, so this is not a continuous function.

17. Two possible graphs are shown in Figures 1.70 and 1.71.

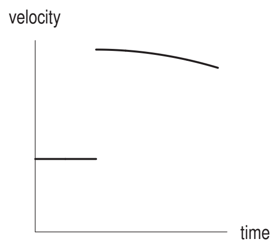


Figure 1.70: Velocity of the car

Figure 1.71: Distance

The distance moved by the car is continuous. (Figure 1.71 has no breaks in it.) In actual fact, the velocity of the car is also continuous; however, in this case, it is well-approximated by the function in Figure 1.70, which is not continuous on any interval containing the moment of impact.

18. The voltage $f(t)$ is graphed in Figure 1.72.

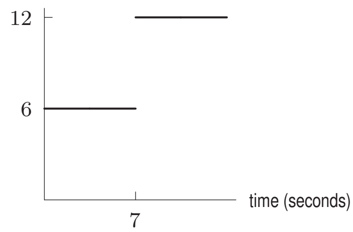


Figure 1.72: Voltage change from 6V to 12V

Using formulas, the voltage, $f(t)$, is represented by

$$f(t) = \begin{cases} 6, & 0 < t \le 7 \\ 12, & 7 < t \end{cases}$$

Although a real physical voltage is continuous, the voltage in this circuit is well-approximated by the function $f(t)$, which is not continuous on any interval around 7 seconds.

19. The value of y on the line $y = kx$ at $x = 3$ is $y = 3k$. To make $f(x)$ continuous, we need

$$3k = 5 \quad \text{so} \quad k = \frac{5}{3}.$$

See Figure 1.73.

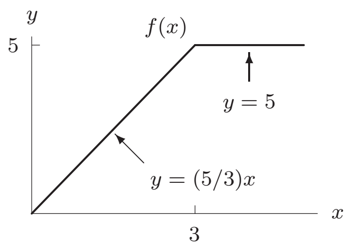


Figure 1.73

20. For any value of k, the function is continuous at every point except $x = 2$. We choose k to make the function continuous at $x = 2$.

Since $3x^2$ takes the value $3(2^2) = 12$ at $x = 2$, we choose k so that the graph of kx goes through the point $(2, 12)$. Thus $k = 6$.

21. If the graphs of $y = t + k$ and $y = kt$ meet at $t = 5$, we have

$$\begin{aligned} 5 + k &= 5k \\ k &= 5/4. \end{aligned}$$

See Figure 1.74.

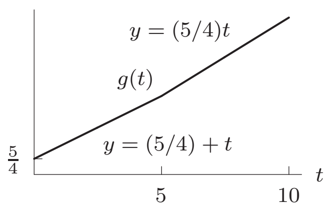


Figure 1.74

22. At $x = \pi$, the curve $y = k\cos x$ has $y = k\cos\pi = -k$. At $x = \pi$, the line $y = 12 - x$ has $y = 12 - \pi$. If $h(x)$ is continuous, we need

$$\begin{aligned} -k &= 12 - \pi \\ k &= \pi - 12. \end{aligned}$$

23. (a) See Figure 1.75.

(b) For any value of k, the function is continuous at every point except $x = 2$. We choose k to make the function continuous at $x = 2$.

Since $(x-2)^2 + 3$ takes on the value $(2-2)^2 + 3 = 3$ at $x = 2$, we choose k so that $kx = 3$ at $x = 2$, so $2k = 3$ and $k = 3/2$.

(c) See Figure 1.76.

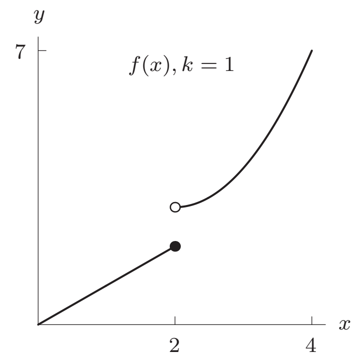


Figure 1.75

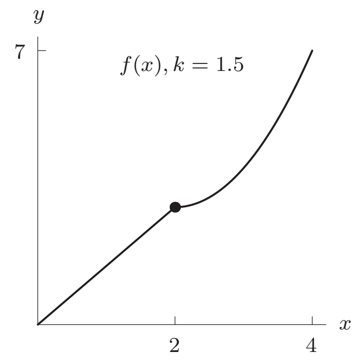


Figure 1.76

24. For any value of k, the function is continuous at every point except $x = 1$. We choose k to make the function continuous at $x = 1$.

Since $x + 3$ takes the value $1 + 3 = 4$ at $x = 1$, we choose k so that the graph of kx goes through the point $(1, 4)$. Thus $k = 4$.

25. For any value of k, the function is continuous at every point except $x = 1$. We choose k to make the function continuous at $x = 1$.

Since kx takes the value $k \cdot 1 = k$ at $x = 1$, we choose k so that the graph of $2kx + 3$ goes through the point $(1, k)$. This gives

$$\begin{aligned} 2k \cdot 1 + 3 &= k \\ k &= -3. \end{aligned}$$

26. For any value of k, the function is continuous at every point except $x = \pi$. We choose k to make the function continuous at $x = \pi$.

Since $k \sin x$ takes the value $k \sin \pi = 0$ at $x = \pi$, we cannot choose k so that the graph of $x + 4$ goes through the point $(\pi, 0)$. Thus, this function is discontinuous for all values of k.

27. For any value of k, the function is continuous at every point except $x = 2$. We choose k to make the function continuous at $x = 2$.

Since $x + 1$ takes the value $2 + 1 = 3$ at $x = 2$, we choose k so that the graph of e^{kx} goes through the point $(2, 3)$. This gives

$$\begin{aligned} e^{2k} &= 3 \\ 2k &= \ln 3 \\ k &= \frac{\ln 3}{2} \end{aligned}$$

28. For any value of k, the function is continuous at every point except $x = 1$. We choose k to make the function continuous at $x = 1$.

Since $\sin(kx)$ takes the value $\sin k$ at $x = 1$, we choose k so that the graph of $0.5x$ goes through the point $(1, \sin k)$. This gives

$$\begin{aligned} \sin k &= 0.5 \\ k &= \sin^{-1} 0.5 = \frac{\pi}{6}. \end{aligned}$$

Other solutions are possible.

29. For any value of k, the function is continuous at every point except $x = 2$. We choose k to make the function continuous at $x = 2$.

Since $\ln(kx + 1)$ takes the value $\ln(2k + 1)$ at $x = 2$, we choose k so that the graph of $x + 4$ goes through the point $(2, \ln(2k + 1))$. This gives

$$\begin{aligned} \ln(2k+1) &= 2+4 = 6 \\ 2k+1 &= e^6 \\ k &= \frac{e^6-1}{2}. \end{aligned}$$

30. (a) The initial value is when $t = 0$, and we see that $P(0) = e^{k\cdot 0} = e^0 = 1000$.
(b) Since the function is continuous, at $t = 12$, we have $e^{kt} = 100$ and we solve for k:

$$\begin{aligned} e^{12k} &= 100 \\ 12k &= \ln 100 \\ k &= \frac{\ln 100}{12} = 0.384. \end{aligned}$$

(c) The population is increasing exponentially for 12 months and then becoming constant.

31. For $x > 0$, we have $|x| = x$, so $f(x) = 1$. For $x < 0$, we have $|x| = -x$, so $f(x) = -1$. Thus, the function is given by

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 1 \end{cases},$$

so f is not continuous on any interval containing $x = 0$.

32. The graph of g suggests that g is not continuous on any interval containing $\theta = 0$, since $g(0) = 1/2$.

33. The drug first increases linearly for half a second, at the end of which time there is 0.6 ml in the body. Thus, for $0 \le t \le 0.5$, the function is linear with slope $0.6/0.5 = 1.2$:

$$Q = 1.2t \quad \text{for} \quad 0 \le t \le 0.5.$$

At $t = 0.5$, we have $Q = 0.6$. For $t > 0.5$, the quantity decays exponentially at a continuous rate of 0.002, so Q has the form

$$Q = Ae^{-0.002t} \qquad 0.5 < t.$$

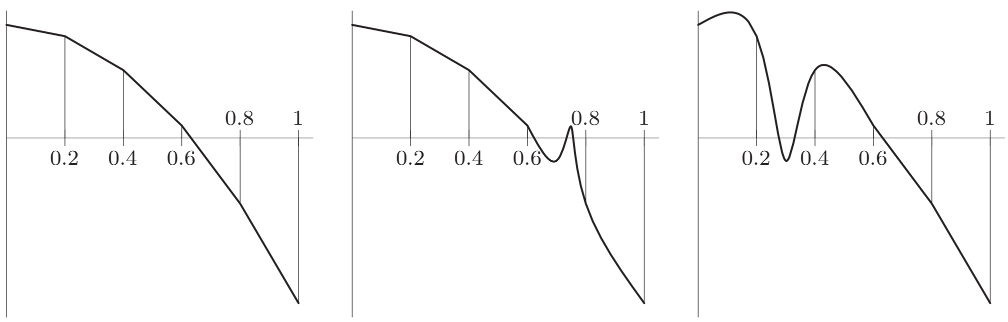
We choose A so that $Q = 0.6$ when $t = 0.5$:

$$\begin{aligned} 0.6 &= Ae^{-0.002(0.5)} = Ae^{-0.001} \\ A &= 0.6e^{0.001}. \end{aligned}$$

Thus

$$Q = \begin{cases} 1.2t & 0 \le t \le 0.5 \\ 0.6e^{0.001}e^{-.002t} & 0.5 < t. \end{cases}$$

34.



35. Since polynomials are continuous, and since $p(5) < 0$ and $p(10) > 0$ and $p(12) < 0$, there are two zeros, one between $x = 5$ and $x = 10$, and another between $x = 10$ and $x = 12$. Thus, $p(x)$ is a cubic with at least two zeros.

If $p(x)$ has only two zeros, one would be a double zero (corresponding to a repeated factor). However, since a polynomial does not change sign at a repeated zero, $p(x)$ cannot have a double zero and have the signs it does.

Thus, $p(x)$ has three zeros. The third zero can be greater than 12 or less than 5. See Figures 1.77 and 1.78.

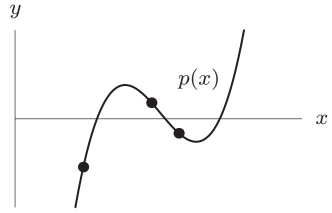


Figure 1.77

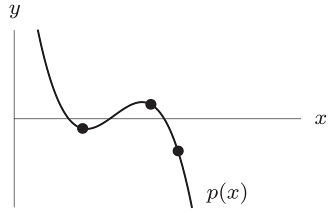


Figure 1.78

36. (a) The graphs of $y = e^x$ and $y = 4 - x^2$ cross twice in Figure 1.79. This tells us that the equation $e^x = 4 - x^2$ has two solutions.

Since $y = e^x$ increases for all x and $y = 4 - x^2$ increases for $x < 0$ and decreases for $x > 0$, these are only the two crossing points.

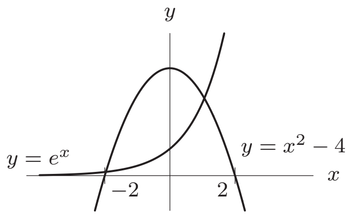


Figure 1.79

(b) Values of $f(x)$ are in Table 1.5. One solution is between $x = -2$ and $x = -1$; the second solution is between $x = 1$ and $x = 2$.

Table 1.5

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	12.0	5.0	0.1	-2.6	-3	-0.3	7.4	25.1	66.6

37. (a) Figure 1.80 shows a possible graph of $f(x)$, yours may be different.

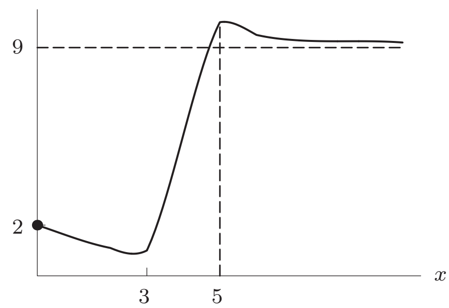


Figure 1.80

(b) In order for f to approach the horizontal asymptote at 9 from above it is necessary that f eventually become concave up. It is therefore not possible for f to be concave down for all $x > 6$.

38. (a) Since $f(x)$ is not continuous at $x = 1$, it does not satisfy the conditions of the Intermediate Value Theorem.
(b) We see that $f(0) = e^0 = 1$ and $f(2) = 4 + (2-1)^2 = 5$. Since e^x is increasing between $x = 0$ and $x = 1$, and since $4 + (x-1)^2$ is increasing between $x = 1$ and $x = 2$, any value of k between $e^1 = e$ and $4 + (1-1)^2 = 4$, such as $k = 3$, is a value such that $f(x) = k$ has no solution.

Strengthen Your Understanding

39. The Intermediate Value theorem only makes this guarantee for a *continuous* function, not for any function.

40. The Intermediate Value Theorem guarantees that for at least one value of x between 0 and 2, we have $f(x) = 5$, but it does not tell us which value(s) of x give $f(x) = 5$.

41. We want a function which has a value at every point but where the graph has a break at $x = 15$. One possibility is

$$f(x) = \begin{cases} 1 & x \geq 15 \\ -1 & x < 15 \end{cases}$$

42. One example is $f(x) = 1/x$, which is not continuous at $x = 0$. The Intermediate Value Theorem does not apply on an interval that contains a point where a function is not continuous.

43. Let $f(x) = \begin{cases} 1 & x \leq 2 \\ x & x > 2 \end{cases}$. Then $f(x)$ is continuous at every point in $[0, 3]$ except at $x = 2$. Other answers are possible.

44. Let $f(x) = \begin{cases} x & x \leq 3 \\ 2x & x > 3 \end{cases}$. Then $f(x)$ is increasing for all x but $f(x)$ is not continuous at $x = 3$. Other answers are possible.

45. False. For example, let $f(x) = \begin{cases} 1 & x \leq 3 \\ 2 & x > 3 \end{cases}$, then $f(x)$ is defined at $x = 3$ but it is not continuous at $x = 3$. (Other examples are possible.)

46. False. A counterexample is graphed in Figure 1.81, in which $f(5) < 0$.

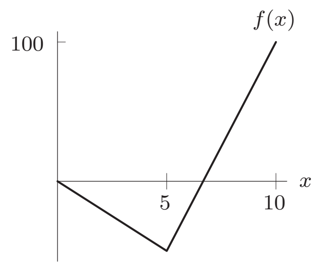


Figure 1.81

47. False. A counterexample is graphed in Figure 1.82.

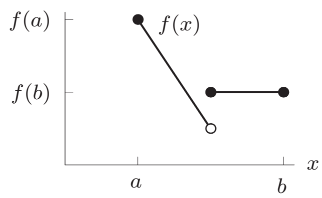


Figure 1.82

Solutions for Section 1.8

Exercises

1. (a) As x approaches -2 from either side, the values of $f(x)$ get closer and closer to 3, so the limit appears to be about 3.
(b) As x approaches 0 from either side, the values of $f(x)$ get closer and closer to 7. (Recall that to find a limit, we are interested in what happens to the function near x but not at x.) The limit appears to be about 7.
(c) As x approaches 2 from either side, the values of $f(x)$ get closer and closer to 3 on one side of $x = 2$ and get closer and closer to 2 on the other side of $x = 2$. Thus the limit does not exist.
(d) As x approaches 4 from either side, the values of $f(x)$ get closer and closer to 8. (Again, recall that we don't care what happens right at $x = 4$.) The limit appears to be about 8.

2. (a) $\lim_{x\to 1^-} f(x) = 1$.
(b) $\lim_{x\to 1^+} f(x)$ does not exist.
(c) $\lim_{x\to 1} f(x)$ does not exist.
(d) $\lim_{x\to 2^-} f(x) = 1$.
(e) $\lim_{x\to 2^+} f(x) = 1$.
(f) $\lim_{x\to 2} f(x) = 1$.

3. From the graphs of f and g, we estimate $\lim_{x\to 1^-} f(x) = 3$, $\lim_{x\to 1^-} g(x) = 5$,
$\lim_{x\to 1^+} f(x) = 4$, $\lim_{x\to 1^+} g(x) = 1$.
(a) $\lim_{x\to 1^-} (f(x) + g(x)) = 3 + 5 = 8$
(b) $\lim_{x\to 1^+} (f(x) + 2g(x)) = \lim_{x\to 1^+} f(x) + 2 \lim_{x\to 1^+} g(x) = 4 + 2(1) = 6$
(c) $\lim_{x\to 1^-} (f(x)g(x)) = (\lim_{x\to 1^-} f(x))(\lim_{x\to 1^-} g(x)) = (3)(5) = 15$
(d) $\lim_{x\to 1^+} (f(x)/g(x)) = \left(\lim_{x\to 1^+} f(x)\right) / \left(\lim_{x\to 1^+} g(x)\right) = 4/1 = 4$

4. We see that $f(x)$ goes to $-\infty$ on both ends, so one possible graph is shown in Figure 1.83. Other answers are possible.

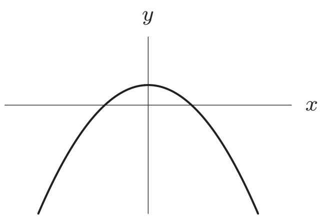


Figure 1.83

5. We see that $f(x)$ goes to $+\infty$ on the left and to $-\infty$ on the right. One possible graph is shown in Figure 1.84. Other answers are possible.

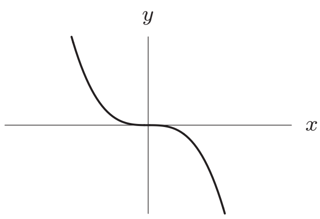


Figure 1.84

6. We see that $f(x)$ goes to $+\infty$ on the left and approaches a y-value of 1 on the right. One possible graph is shown in Figure 1.85. Other answers are possible.

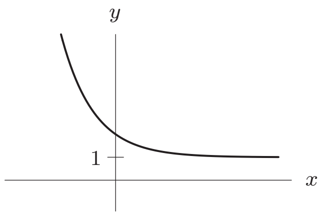


Figure 1.85

7. We see that $f(x)$ approaches a y-value of 3 on the left and goes to $-\infty$ on the right. One possible graph is shown in Figure 1.86. Other answers are possible.

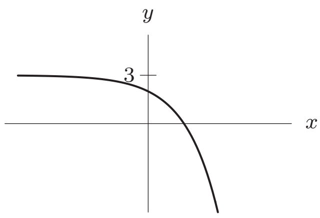


Figure 1.86

8. We see that $f(x)$ goes to $+\infty$ on the right and that it also passes through the point $(-1, 2)$. (Notice that this must be a point on the graph since the instructions require that $f(x)$ be defined and continuous.) One possible graph is shown in Figure 1.87. Other answers are possible.

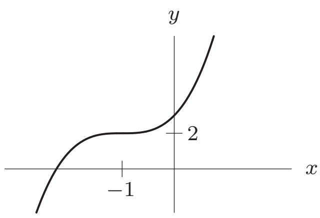


Figure 1.87

9. We see that $f(x)$ goes to $+\infty$ on the left and that it also passes through the point $(3, 5)$. (Notice that this must be a point on the graph since the instructions require that $f(x)$ be defined and continuous.) One possible graph is shown in Figure 1.88. Other answers are possible.

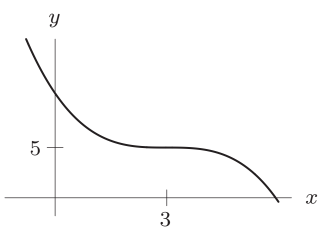


Figure 1.88

10. We see that $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to+\infty} f(x) = -\infty$.

11. As $x \to \pm\infty$, we know that $f(x)$ behaves like its leading term $-2x^3$. Thus, we have $\lim_{x\to-\infty} f(x) = +\infty$ and $\lim_{x\to+\infty} f(x) = -\infty$.

12. As $x \to \pm\infty$, we know that $f(x)$ behaves like its leading term x^5. Thus, we have $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to+\infty} f(x) = +\infty$.

13. As $x \to \pm\infty$, we know that $f(x)$ behaves like the quotient of the leading terms of its numerator and denominator. Since

$$f(x) \to \frac{3x^3}{5x^3} = \frac{3}{5},$$

we have $\lim_{x\to-\infty} f(x) = 3/5$ and $\lim_{x\to+\infty} f(x) = 3/5$.

14. As $x \to \pm\infty$, we know that x^{-3} gets closer and closer to zero, so we have $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to+\infty} f(x) = 0$.

15. We have $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to+\infty} f(x) = +\infty$.

16. The break in the graph at $x = 0$ suggests that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist. See Figure 1.89.

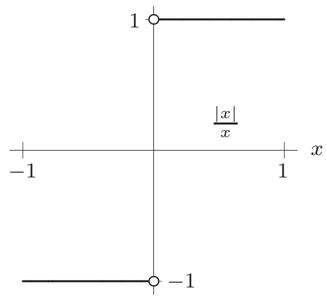


Figure 1.89

17. For $-1 \le x \le 1, \ -1 \le y \le 1$, the graph of $y = x \ln |x|$ is in Figure 1.90. The graph suggests that

$$\lim_{x\to 0} x \ln |x| = 0.$$

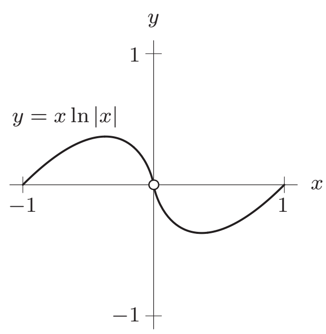


Figure 1.90

18. Since $\lim_{x\to 0^+} \frac{|x|}{x} = 1$ and $\lim_{x\to 0^-} \frac{|x|}{x} = -1$, we say that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist. In addition $f(x)$ is not defined at 0. Therefore, $f(x)$ is not continuous on any interval containing 0.

19. For $-0.5 \leq \theta \leq 0.5$, $0 \leq y \leq 3$, the graph of $y = \dfrac{\sin(2\,\theta)}{\theta}$ is shown in Figure 1.91. Therefore, $\lim_{\theta \to 0} \dfrac{\sin(2\,\theta)}{\theta} = 2$.

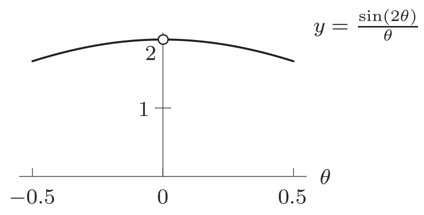


Figure 1.91

20. For $-1 \leq \theta \leq 1$, $-1 \leq y \leq 1$, the graph of $y = \dfrac{\cos\theta - 1}{\theta}$ is shown in Figure 1.92. Therefore, $\lim_{\theta \to 0} \dfrac{\cos\theta - 1}{\theta} = 0$.

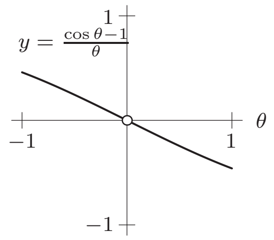


Figure 1.92

21. For $-90^\circ \leq \theta \leq 90^\circ$, $0 \leq y \leq 0.02$, the graph of $y = \dfrac{\sin\theta}{\theta}$ is shown in Figure 1.93. Therefore, by tracing along the curve, we see that in degrees, $\lim_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 0.01745\ldots$.

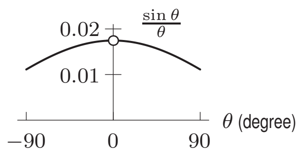


Figure 1.93

22. For $-0.5 \leq \theta \leq 0.5$, $0 \leq y \leq 0.5$, the graph of $y = \dfrac{\theta}{\tan(3\theta)}$ is shown in Figure 1.94. Therefore, by tracing along the curve, we see that $\lim_{\theta \to 0} \dfrac{\theta}{\tan(3\theta)} = 0.3333\ldots$.

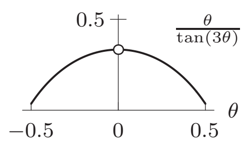


Figure 1.94

23. A graph of $y = \dfrac{e^h - 1}{h}$ in a window such as $-0.5 \leq h \leq 0.5$ and $0 \leq y \leq 3$ appears to indicate $y \to 1$ as $h \to 0$. Therefore, we estimate that $\lim_{h \to 0} \dfrac{e^h - 1}{h} = 1$.

24. A graph of $y = \dfrac{e^{5h}-1}{h}$ in a window such as $-0.5 \le h \le 0.5$ and $0 \le y \le 6$ appears to indicate $y \to 5$ as $h \to 0$. Therefore, we estimate that $\lim_{h\to 0} \dfrac{e^{5h}-1}{h} = 5$.

25. A graph of $y = \dfrac{2^h - 1}{h}$ in a window such as $-0.5 \le h \le 0.5$ and $0 \le y \le 1$ appears to indicate $y \to 0.7$ as $h \to 0$. By zooming in on the graph, we can estimate the limit more accurately. Therefore, we estimate that

$$\lim_{h\to 0} \frac{2^h-1}{h} = 0.693.$$

26. A graph of $y = \dfrac{3^h - 1}{h}$ in a window such as $-0.5 \le h \le 0.5$ and $0 \le y \le 1.5$ appears to indicate $y \to 1.1$ as $h \to 0$. By zooming in on the graph, we can estimate the limit more accurately. Therefore, we estimate that

$$\lim_{h\to 0} \frac{3^h-1}{h} = 1.098.$$

27. A graph of $y = \dfrac{\cos(3h)-1}{h}$ in a window such as $-0.5 \le h \le 0.5$ and $-1 \le y \le 1$ appears to indicate $y \to 0$ as $h \to 0$. Therefore, we estimate that

$$\lim_{h\to 0} \frac{\cos(3h)-1}{h} = 0.$$

28. A graph of $y = \dfrac{\sin(3h)}{h}$ in a window such as $-0.5 \le h \le 0.5$ and $0 \le y \le 4$ appears to indicate $y \to 3$ as $h \to 0$. Therefore, we estimate that

$$\lim_{h\to 0} \frac{\sin(3h)}{h} = 3.$$

29. $f(x) = \dfrac{|x-4|}{x-4} = \begin{cases} \dfrac{x-4}{x-4} & x > 4 \\ -\dfrac{x-4}{x-4} & x < 4 \end{cases} = \begin{cases} 1 & x > 4 \\ -1 & x < 4 \end{cases}$

Figure 1.95 confirms that $\lim_{x\to 4^+} f(x) = 1$, $\lim_{x\to 4^-} f(x) = -1$ so $\lim_{x\to 4} f(x)$ does not exist.

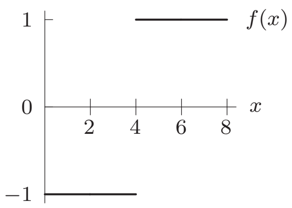


Figure 1.95

30. $f(x) = \dfrac{|x-2|}{x} = \begin{cases} \dfrac{x-2}{x}, & x > 2 \\ -\dfrac{x-2}{x}, & x < 2 \end{cases}$

Figure 1.96 confirms that $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = \lim_{x\to 2} f(x) = 0$.

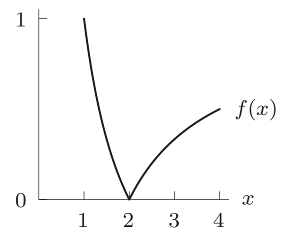


Figure 1.96

31. $f(x) = \begin{cases} x^2 - 2 & 0 < x < 3 \\ 2 & x = 3 \\ 2x + 1 & 3 < x \end{cases}$

Figure 1.97 confirms that $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x^2 - 2) = 7$ and that $\lim_{x\to 3+} f(x) = \lim_{x\to 3+} (2x+1) = 7$, so $\lim_{x\to 3} f(x) = 7$. Note, however, that $f(x)$ is not continuous at $x = 3$ since $f(3) = 2$.

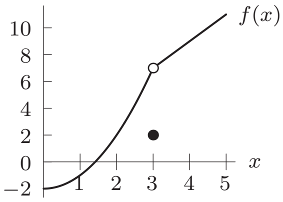


Figure 1.97

32. The graph in Figure 1.98 suggests that

$$\text{if } -0.05 < \theta < 0.05, \quad \text{then} \quad 0.999 < (\sin\theta)/\theta < 1.001.$$

Thus, if θ is within 0.05 of 0, we see that $(\sin\theta)/\theta$ is within 0.001 of 1.

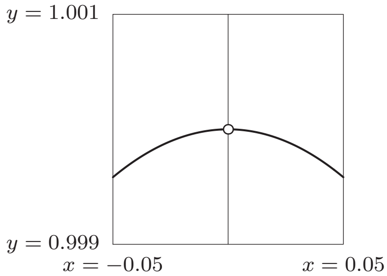


Figure 1.98: Graph of $(\sin\theta)/\theta$ with $-0.05 < \theta < 0.05$

33. The statement

$$\lim_{h\to a} g(h) = K$$

means that we can make the value of $g(h)$ as close to K as we want by choosing h sufficiently close to, but not equal to, a.

In symbols, for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$|g(h) - K| < \epsilon \quad \text{for all } 0 < |h - a| < \delta.$$

Problems

34. At $x = 0$, the function is not defined. In addition, $\lim_{x\to 0} f(x)$ does not exist. Thus, $f(x)$ is not continuous at $x = 0$.

35. Since $\lim_{x\to 0^+} f(x) = 1$ and $\lim_{x\to 0^-} f(x) = -1$, we see that $\lim_{x\to 0} f(x)$ does not exist. Thus, $f(x)$ is not continuous at $x = 0$

36. Since $x/x = 1$ for $x \neq 0$, this function $f(x) = 1$ for all x. Thus, $f(x)$ is continuous for all x.

37. Since $2x/x = 2$ for $x \neq 0$, we have $\lim_{x\to 0} f(x) = 2$, so

$$\lim_{x\to 0} f(x) \neq f(0) = 3.$$

Thus, $f(x)$ is not continuous at $x = 0$.

38. The answer (see the graph in Figure 1.99) appears to be about 2.7; if we zoom in further, it appears to be about 2.72, which is close to the value of $e \approx 2.71828$.

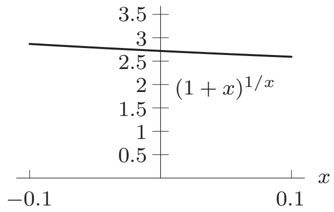


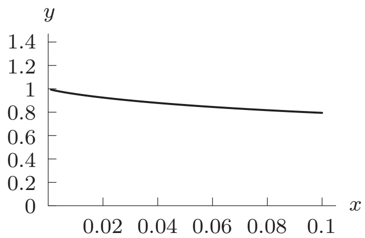
Figure 1.99

39. We use values of h approaching, but not equal to, zero. If we let $h = 0.01, 0.001, 0.0001, 0.00001$, we calculate the values 2.7048, 2.7169, 2.7181, and 2.7183. If we let $h = -0.01$ -0.001, -0.0001, -0.00001, we get values 2.7320, 2.7196, 2.7184, and 2.7183. These numbers suggest that the limit is the number $e = 2.71828\ldots$. However, these calculations cannot tell us that the limit is exactly e; for that a proof is needed.

40. When $x = 0.1$, we find $xe^{1/x} \approx 2203$. When $x = 0.01$, we find $xe^{1/x} \approx 3 \times 10^{41}$. When $x = 0.001$, the value of $xe^{1/x}$ is too big for a calculator to compute. This suggests that $\lim_{x \to 0^+} xe^{1/x}$ does not exist (and in fact it does not).

41. If $x > 1$ and x approaches 1, then $p(x) = 55$. If $x < 1$ and x approaches 1, then $p(x) = 34$. There is not a single number that $p(x)$ approaches as x approaches 1, so we say that $\lim_{x \to 1} p(x)$ does not exist.

42. The limit appears to be 1; a graph and table of values is shown below.



x	x^x
0.1	0.7943
0.01	0.9550
0.001	0.9931
0.0001	0.9990
0.00001	0.9999

43. The only change is that, instead of considering all x near c, we only consider x near to and greater than c. Thus the phrase "$|x - c| < \delta$" must be replaced by "$c < x < c + \delta$." Thus, we define

$$\lim_{x \to c^+} f(x) = L$$

to mean that for any $\epsilon > 0$ (as small as we want), there is a $\delta > 0$ (sufficiently small) such that if $c < x < c + \delta$, then $|f(x) - L| < \epsilon$.

44. The only change is that, instead of considering all x near c, we only consider x near to and less than c. Thus the phrase "$|x - c| < \delta$" must be replaced by "$c - \delta < x < c$." Thus, we define

$$\lim_{x \to c^-} f(x) = L$$

to mean that for any $\epsilon > 0$ (as small as we want), there is a $\delta > 0$ (sufficiently small) such that if $c - \delta < x < c$, then $|f(x) - L| < \epsilon$.

45. Instead of being "sufficiently close to c," we want x to be "sufficiently large." Using N to measure how large x must be, we define

$$\lim_{x \to \infty} f(x) = L$$

to mean that for any $\epsilon > 0$ (as small as we want), there is a $N > 0$ (sufficiently large) such that if $x > N$, then $|f(x) - L| < \epsilon$.

46. From Table 1.6, it appears the limit is 1. This is confirmed by Figure 1.100. An appropriate window is $-0.0033 < x < 0.0033$, $0.99 < y < 1.01$.

Table 1.6

x	$f(x)$
0.1	1.3
0.01	1.03
0.001	1.003
0.0001	1.0003

x	$f(x)$
−0.0001	0.9997
−0.001	0.997
−0.01	0.97
−0.1	0.7

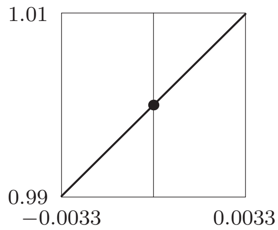


Figure 1.100

47. From Table 1.7, it appears the limit is −1. This is confirmed by Figure 1.101. An appropriate window is $-0.099 < x < 0.099$, $-1.01 < y < -0.99$.

Table 1.7

x	$f(x)$
0.1	−0.99
0.01	−0.9999
0.001	−0.999999
0.0001	−0.99999999

x	$f(x)$
−0.0001	−0.99999999
−0.001	−0.999999
−0.01	−0.9999
−0.1	−0.99

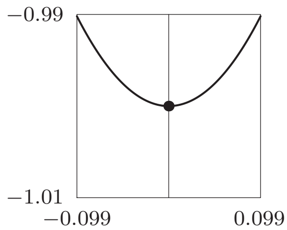


Figure 1.101

48. From Table 1.8, it appears the limit is 0. This is confirmed by Figure 1.102. An appropriate window is $-0.005 < x < 0.005$, $-0.01 < y < 0.01$.

Table 1.8

x	$f(x)$
0.1	0.1987
0.01	0.0200
0.001	0.0020
0.0001	0.0002

x	$f(x)$
−0.0001	−0.0002
−0.001	−0.0020
−0.01	−0.0200
−0.1	−0.1987

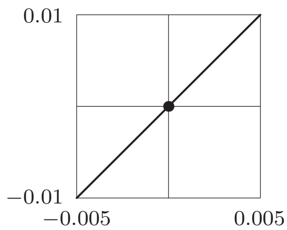


Figure 1.102

49. From Table 1.9, it appears the limit is 0. This is confirmed by Figure 1.103. An appropriate window is $-0.0033 < x < 0.0033$, $-0.01 < y < 0.01$.

Table 1.9

x	$f(x)$
0.1	0.2955
0.01	0.0300
0.001	0.0030
0.0001	0.0003

x	$f(x)$
−0.0001	−0.0003
−0.001	−0.0030
−0.01	−0.0300
−0.1	−0.2955

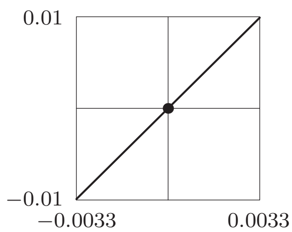


Figure 1.103

50. From Table 1.10, it appears the limit is 2. This is confirmed by Figure 1.104. An appropriate window is $-0.0865 < x < 0.0865$, $1.99 < y < 2.01$.

Table 1.10

x	$f(x)$
0.1	1.9867
0.01	1.9999
0.001	2.0000
0.0001	2.0000

x	$f(x)$
−0.0001	2.0000
−0.001	2.0000
−0.01	1.9999
−0.1	1.9867

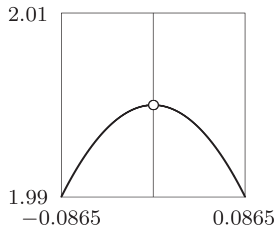


Figure 1.104

51. From Table 1.11, it appears the limit is 3. This is confirmed by Figure 1.105. An appropriate window is $-0.047 < x < 0.047$, $2.99 < y < 3.01$.

Table 1.11

x	$f(x)$
0.1	2.9552
0.01	2.9996
0.001	3.0000
0.0001	3.0000

x	$f(x)$
−0.0001	3.0000
−0.001	3.0000
−0.01	2.9996
−0.1	2.9552

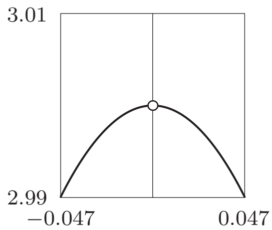


Figure 1.105

52. From Table 1.12, it appears the limit is 1. This is confirmed by Figure 1.106. An appropriate window is $-0.0198 < x < 0.0198$, $0.99 < y < 1.01$.

Table 1.12

x	$f(x)$
0.1	1.0517
0.01	1.0050
0.001	1.0005
0.0001	1.0001

x	$f(x)$
−0.0001	1.0000
−0.001	0.9995
−0.01	0.9950
−0.1	0.9516

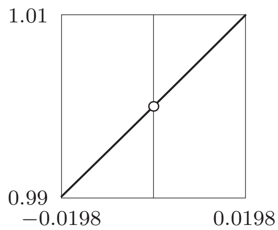


Figure 1.106

53. From Table 1.13, it appears the limit is 2. This is confirmed by Figure 1.107. An appropriate window is $-0.0049 < x < 0.0049$, $1.99 < y < 2.01$.

Table 1.13

x	$f(x)$
0.1	2.2140
0.01	2.0201
0.001	2.0020
0.0001	2.0002

x	$f(x)$
−0.0001	1.9998
−0.001	1.9980
−0.01	1.9801
−0.1	1.8127

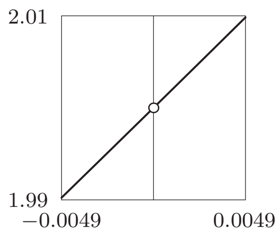


Figure 1.107

54. Divide numerator and denominator by x:

$$f(x) = \frac{x+3}{2-x} = \frac{1+3/x}{2/x-1},$$

so

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1+3/x}{2/x-1} = \frac{\lim_{x\to\infty}(1+3/x)}{\lim_{x\to\infty}(2/x-1)} = \frac{1}{-1} = -1.$$

55. Divide numerator and denominator by x:

$$f(x) = \frac{\pi+3x}{\pi x-3} = \frac{(\pi+3x)/x}{(\pi x-3)/x},$$

so

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\pi/x+3}{\pi-3/x} = \frac{\lim_{x\to\infty}(\pi/x+3)}{\lim_{x\to\infty}(\pi-3/x)} = \frac{3}{\pi}.$$

56. Divide numerator and denominator by x^2:

$$f(x) = \frac{x-5}{5+2x^2} = \frac{(1/x)-(5/x^2)}{(5/x^2)+2},$$

so

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{(1/x)-(5/x^2)}{(5/x^2)+2} = \frac{\lim_{x\to\infty}((1/x)-(5/x^2))}{\lim_{x\to\infty}((5/x^2)+2)} = \frac{0}{2} = 0.$$

57. Divide numerator and denominator by x^2, giving

$$f(x) = \frac{x^2+2x-1}{3+3x^2} = \frac{1+2/x-1/x^2}{3/x^2+3},$$

so

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1+2/x-1/x^2}{3/x^2+3} = \frac{\lim_{x\to\infty}(1+2/x-1/x^2)}{\lim_{x\to\infty}(3/x^2+3)} = \frac{1}{3}.$$

58. Divide numerator and denominator by x, giving

$$f(x) = \frac{x^2+4}{x+3} = \frac{x+4/x}{1+3/x},$$

so

$$\lim_{x\to\infty} f(x) = +\infty.$$

59. Divide numerator and denominator by x^3, giving

$$f(x) = \frac{2x^3-16x^2}{4x^2+3x^3} = \frac{2-16/x}{4/x+3},$$

so

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2-16/x}{4/x+3} = \frac{\lim_{x\to\infty}(2-16/x)}{\lim_{x\to\infty}(4/x+3)} = \frac{2}{3}.$$

60. Divide numerator and denominator by x^5, giving

$$f(x) = \frac{x^4+3x}{x^4+2x^5} = \frac{1/x+3/x^4}{1/x+2},$$

so

$$\lim_{x\to\infty} f(x) = \frac{\lim_{x\to\infty}(1/x+3/x^4)}{\lim_{x\to\infty}(1/x+2)} = \frac{0}{2} = 0.$$

61. Divide numerator and denominator by e^x, giving

$$f(x) = \frac{3e^x + 2}{2e^x + 3} = \frac{3 + 2e^{-x}}{2 + 3e^{-x}},$$

so

$$\lim_{x\to\infty} f(x) = \frac{\lim_{x\to\infty}(3 + 2e^{-x})}{\lim_{x\to\infty}(2 + 3e^{-x})} = \frac{3}{2}.$$

62. Since $\lim_{x\to\infty} 2^{-x} = \lim_{x\to\infty} 3^{-x} = 0$, we have

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2^{-x} + 5}{3^{-x} + 7} = \frac{\lim_{x\to\infty}(2^{-x} + 5)}{\lim_{x\to\infty}(3^{-x} + 7)} = \frac{5}{7}.$$

63. $f(x) = \dfrac{2e^{-x} + 3}{3e^{-x} + 2}$, so $\lim\limits_{x\to\infty} f(x) = \dfrac{\lim_{x\to\infty}(2e^{-x} + 3)}{\lim_{x\to\infty}(3e^{-x} + 2)} = \dfrac{3}{2}.$

64. Because the denominator equals 0 when $x = 4$, so must the numerator. This means $k^2 = 16$ and the choices for k are 4 or -4.

65. Because the denominator equals 0 when $x = 1$, so must the numerator. So $1 - k + 4 = 0$. The only possible value of k is 5.

66. Because the denominator equals 0 when $x = -2$, so must the numerator. So $4 - 8 + k = 0$ and the only possible value of k is 4.

67. Division of numerator and denominator by x^2 yields

$$\frac{x^2 + 3x + 5}{4x + 1 + x^k} = \frac{1 + 3/x + 5/x^2}{4/x + 1/x^2 + x^{k-2}}.$$

As $x \to \infty$, the limit of the numerator is 1. The limit of the denominator depends upon k. If $k > 2$, the denominator approaches ∞ as $x \to \infty$, so the limit of the quotient is 0. If $k = 2$, the denominator approaches 1 as $x \to \infty$, so the limit of the quotient is 1. If $k < 2$ the denominator approaches 0^+ as $x \to \infty$, so the limit of the quotient is ∞. Therefore the values of k we are looking for are $k \geq 2$.

68. For the numerator, $\lim\limits_{x\to-\infty} \left(e^{2x} - 5\right) = -5$. If $k > 0$, $\lim\limits_{x\to-\infty} \left(e^{kx} + 3\right) = 3$, so the quotient has a limit of $-5/3$. If $k = 0$, $\lim\limits_{x\to-\infty} \left(e^{kx} + 3\right) = 4$, so the quotient has limit of $-5/4$. If $k < 0$, the limit of the quotient is given by $\lim\limits_{x\to-\infty} (e^{2x} - 5)/(e^{kx} + 3) = 0$.

69. Division of numerator and denominator by x^3 yields

$$\frac{x^3 - 6}{x^k + 3} = \frac{1 - 6/x^3}{x^{k-3} + 3/x^3}.$$

As $x \to \infty$, the limit of the numerator is 1. The limit of the denominator depends upon k. If $k > 3$, the denominator approaches ∞ as $x \to \infty$, so the limit of the quotient is 0. If $k = 3$, the denominator approaches 1 as $x \to \infty$, so the limit of the quotient is 1. If $k < 3$ the denominator approaches 0^+ as $x \to \infty$, so the limit of the quotient is ∞. Therefore the values of k we are looking for are $k \geq 3$.

70. We divide both the numerator and denominator by 3^{2x}, giving

$$\lim_{x\to\infty} \frac{3^{kx} + 6}{3^{2x} + 4} = \frac{3^{(k-2)x} + 6/3^{2x}}{1 + 4/3^{2x}}.$$

In the denominator, $\lim\limits_{x\to\infty} 1 + 4/3^{2x} = 1$. In the numerator, if $k < 2$, we have $\lim\limits_{x\to\infty} 3^{(k-2)x} + 6/3^{2x} = 0$, so the quotient has a limit of 0. If $k = 2$, we have $\lim\limits_{x\to\infty} 3^{(k-2)x} + 6/3^{2x} = 1$, so the quotient has a limit of 1. If $k > 2$, we have $\lim\limits_{x\to\infty} 3^{(k-2)x} + 6/3^{2x} = \infty$, so the quotient has a limit of ∞.

71. In the denominator, we have $\lim\limits_{x\to-\infty} 3^{2x} + 4 = 4$. In the numerator, if $k < 0$, we have $\lim\limits_{x\to-\infty} 3^{kx} + 6 = \infty$, so the quotient has a limit of ∞. If $k = 0$, we have $\lim\limits_{x\to-\infty} 3^{kx} + 6 = 7$, so the quotient has a limit of $7/4$. If $k > 0$, we have $\lim\limits_{x\to-\infty} 3^{kx} + 6 = 6$, so the quotient has a limit of $6/4$.

72. By tracing on a calculator or solving equations, we find the following values of δ:
For $\epsilon = 0.2$, $\delta \leq 0.1$.
For $\epsilon = 0.1$, $\delta \leq 0.05$.
For $\epsilon = 0.02$, $\delta \leq 0.01$.
For $\epsilon = 0.01$, $\delta \leq 0.005$.
For $\epsilon = 0.002$, $\delta \leq 0.001$.
For $\epsilon = 0.001$, $\delta \leq 0.0005$.

73. By tracing on a calculator or solving equations, we find the following values of δ:
For $\epsilon = 0.1$, $\delta \leq 0.46$.
For $\epsilon = 0.01$, $\delta \leq 0.21$.
For $\epsilon = 0.001$, $\delta < 0.1$. Thus, we can take $\delta \leq 0.09$.

74. The results of Problem 72 suggest that we can choose $\delta = \epsilon/2$. For any $\epsilon > 0$, we want to find the δ such that

$$|f(x) - 3| = |-2x + 3 - 3| = |2x| < \epsilon.$$

Then if $|x| < \delta = \epsilon/2$, it follows that $|f(x) - 3| = |2x| < \epsilon$. So $\lim_{x\to 0}(-2x + 3) = 3$.

75. (a) Since $\sin(n\pi) = 0$ for $n = 1, 2, 3, \ldots$ the sequence of x-values

$$\frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \ldots$$

works. These x-values $\to 0$ and are zeroes of $f(x)$.

(b) Since $\sin(n\pi/2) = 1$ for $n = 1, 5, 9 \ldots$ the sequence of x-values

$$\frac{2}{\pi}, \frac{2}{5\pi}, \frac{2}{9\pi}, \ldots$$

works.

(c) Since $\sin(n\pi)/2 = -1$ for $n = 3, 7, 11, \ldots$ the sequence of x-values

$$\frac{2}{3\pi}, \frac{2}{7\pi}, \frac{2}{11\pi} \ldots$$

works.

(d) Any two of these sequences of x-values show that if the limit were to exist, then it would have to have two (different) values: 0 and 1, or 0 and -1, or 1 and -1. Hence, the limit can not exist.

76. From Table 1.14, it appears the limit is 0. This is confirmed by Figure 1.108. An appropriate window is $-0.015 < x < 0.015$, $-0.01 < y < 0.01$.

Table 1.14

x	$f(x)$
0.1	0.0666
0.01	0.0067
0.001	0.0007
0.0001	0.0001

x	$f(x)$
-0.0001	-0.0001
-0.001	-0.0007
-0.01	-0.0067
-0.1	-0.0666

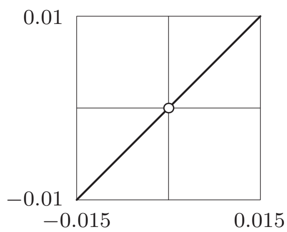


Figure 1.108

77. From Table 1.15, it appears the limit is 0. This is confirmed by Figure 1.109. An appropriate window is $-0.0029 < x < 0.0029$, $-0.01 < y < 0.01$.

Table 1.15

x	$f(x)$
0.1	0.3365
0.01	0.0337
0.001	0.0034
0.0001	0.0004

x	$f(x)$
-0.0001	-0.0004
-0.001	-0.0034
-0.01	-0.0337
-0.1	-0.3365

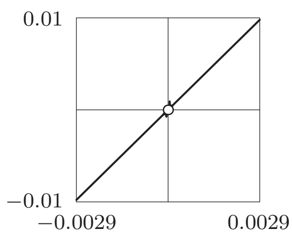


Figure 1.109

78. (a) If $b = 0$, then the property says $\lim_{x\to c} 0 = 0$, which is easy to see is true.

(b) If $|f(x) - L| < \frac{\epsilon}{|b|}$, then multiplying by $|b|$ gives

$$|b||f(x) - L| < \epsilon.$$

Since

$$|b||f(x) - L| = |b(f(x) - L)| = |bf(x) - bL|,$$

we have

$$|bf(x) - bL| < \epsilon.$$

(c) Suppose that $\lim_{x\to c} f(x) = L$. We want to show that $\lim_{x\to c} bf(x) = bL$. If we are to have

$$|bf(x) - bL| < \epsilon,$$

then we will need

$$|f(x) - L| < \frac{\epsilon}{|b|}.$$

We choose δ small enough that

$$|x - c| < \delta \quad \text{implies} \quad |f(x) - L| < \frac{\epsilon}{|b|}.$$

By part (b), this ensures that

$$|bf(x) - bL| < \epsilon,$$

as we wanted.

79. Suppose $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} g(x) = L_2$. Then we need to show that

$$\lim_{x\to c} (f(x) + g(x)) = L_1 + L_2.$$

Let $\epsilon > 0$ be given. We need to show that we can choose $\delta > 0$ so that whenever $|x - c| < \delta$, we will have $|(f(x) + g(x)) - (L_1 + L_2)| < \epsilon$. First choose $\delta_1 > 0$ so that $|x - c| < \delta_1$ implies $|f(x) - L_1| < \frac{\epsilon}{2}$; we can do this since $\lim_{x\to c} f(x) = L_1$. Similarly, choose $\delta_2 > 0$ so that $|x - c| < \delta_2$ implies $|g(x) - L_2| < \frac{\epsilon}{2}$. Now, set δ equal to the smaller of δ_1 and δ_2. Thus $|x - c| < \delta$ will make both $|x - c| < \delta_1$ and $|x - c| < \delta_2$. Then, for $|x - c| < \delta$, we have

$$\begin{aligned} |f(x) + g(x) - (L_1 + L_2)| &= |(f(x) - L_1) + (g(x) - L_2)| \\ &\le |(f(x) - L_1)| + |(g(x) - L_2)| \\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This proves $\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$, which is the result we wanted to prove.

80. (a) We need to show that for any given $\epsilon > 0$, there is a $\delta > 0$ so that $|x - c| < \delta$ implies $|f(x)g(x)| < \epsilon$. If $\epsilon > 0$ is given, choose δ_1 so that when $|x - c| < \delta_1$, we have $|f(x)| < \sqrt{\epsilon}$. This can be done since $\lim_{x\to 0} f(x) = 0$. Similarly, choose δ_2 so that when $|x - c| < \delta_2$, we have $|g(x)| < \sqrt{\epsilon}$. Then, if we take δ to be the smaller of δ_1 and δ_2, we'll have that $|x - c| < \delta$ implies both $|f(x)| < \sqrt{\epsilon}$ and $|g(x)| < \sqrt{\epsilon}$. So when $|x - c| < \delta$, we have $|f(x)g(x)| = |f(x)|\,|g(x)| < \sqrt{\epsilon} \cdot \sqrt{\epsilon} = \epsilon$. Thus $\lim_{x\to c} f(x)g(x) = 0$.

(b) $(f(x) - L_1)(g(x) - L_2) + L_1 g(x) + L_2 f(x) - L_1 L_2$
$= f(x)g(x) - L_1 g(x) - L_2 f(x) + L_1 L_2 + L_1 g(x) + L_2 f(x) - L_1 L_2 = f(x)g(x).$

(c) $\lim_{x\to c} (f(x) - L_1) = \lim_{x\to c} f(x) - \lim_{x\to c} L_1 = L_1 - L_1 = 0$, using the second limit property. Similarly, $\lim_{x\to c} (g(x) - L_2) = 0$.

(d) Since $\lim_{x\to c} (f(x) - L_1) = \lim_{x\to c} (g(x) - L_2) = 0$, we have that $\lim_{x\to c} (f(x) - L_1)(g(x) - L_2) = 0$ by part (a).

(e) From part (b), we have

$$\begin{aligned} \lim_{x\to c} f(x)g(x) &= \lim_{x\to c} ((f(x) - L_1)(g(x) - L_2) + L_1 g(x) + L_2 f(x) - L_1 L_2) \\ &= \lim_{x\to c} (f(x) - L_1)(g(x) - L_2) + \lim_{x\to c} L_1 g(x) + \lim_{x\to c} L_2 f(x) + \lim_{x\to c} (-L_1 L_2) \\ &\quad \text{(using limit property 2)} \\ &= 0 + L_1 \lim_{x\to c} g(x) + L_2 \lim_{x\to c} f(x) - L_1 L_2 \\ &\quad \text{(using limit property 1 and part (d))} \\ &= L_1 L_2 + L_2 L_1 - L_1 L_2 = L_1 L_2. \end{aligned}$$

81. We will show $f(x) = x$ is continuous at $x = c$. Since $f(c) = c$, we need to show that

$$\lim_{x\to c} f(x) = c$$

that is, since $f(x) = x$, we need to show

$$\lim_{x\to c} x = c.$$

Pick any $\epsilon > 0$, then take $\delta = \epsilon$. Thus,

$$|f(x) - c| = |x - c| < \epsilon \quad \text{for all} \quad |x - c| < \delta = \epsilon.$$

82. Since $f(x) = x$ is continuous, Theorem 1.3 on page 64 shows that products of the form $f(x) \cdot f(x) = x^2$ and $f(x) \cdot x^2 = x^3$, etc., are continuous. By a similar argument, x^n is continuous for any $n > 0$.

83. If c is in the interval, we know $\lim\limits_{x\to c} f(x) = f(c)$ and $\lim_{x\to c} g(x) = g(c)$. Then,

$$\begin{aligned}\lim_{x\to c} (f(x) + g(x)) &= \lim_{x\to c} f(x) + \lim_{x\to c} g(x) \quad \text{by limit property 2} \\ &= f(c) + g(c), \quad \text{so } f + g \text{ is continuous at } x = c.\end{aligned}$$

Also,

$$\begin{aligned}\lim_{x\to c} (f(x)g(x)) &= \lim_{x\to c} f(x) \lim_{x\to c} g(x) \quad \text{by limit property 3} \\ &= f(c)g(c) \quad \text{so } fg \text{ is continuous at } x = c.\end{aligned}$$

Finally,

$$\begin{aligned}\lim_{x\to c} \frac{f(x)}{g(x)} &= \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} \quad \text{by limit property 4} \\ &= \frac{f(c)}{g(c)}, \quad \text{so } \frac{f}{g} \text{ is continuous at } x = c.\end{aligned}$$

Strengthen Your Understanding

84. Though $P(x)$ and $Q(x)$ are both continuous for all x, it is possible for $Q(x)$ to be equal to zero for some x. For any such value of x, where $Q(x) = 0$, we see that $P(x)/Q(x)$ is undefined, and thus not continuous. For example,

$$\frac{P(x)}{Q(x)} = \frac{x}{x-1}$$

is not defined or continuous at $x = 1$.

85. The left- and right-hand limits are not the same:

$$\lim_{x\to 1^-} \frac{x-1}{|x-1|} = -1,$$

but

$$\lim_{x\to 1^+} \frac{x-1}{|x-1|} = 1.$$

Since the left- and right-hand limits are not the same, the limit does not exist and thus is not equal to 1.

86. For f to be continuous at $x = c$, we need $\lim_{x\to c} f(x)$ to exist and to be equal to $f(c)$.

87. One possibility is

$$f(x) = \frac{(x+3)(x-1)}{x-1}.$$

We have $\lim_{x\to 1} f(x) = 4$ but $f(1)$ is undefined.

88. One example is

$$f(x) = \frac{2|x|}{x}.$$

89. True, by Property 3 of limits in Theorem 1.2, since $\lim_{x\to 3} x = 3$.

90. False. If $\lim_{x\to 3} g(x)$ does not exist, then $\lim_{x\to 3} f(x)g(x)$ may not even exist. For example, let $f(x) = 2x + 1$ and define g by:

$$g(x) = \begin{cases} 1/(x-3) & \text{if } x \neq 3 \\ 4 & \text{if } x = 3 \end{cases}$$

Then $\lim_{x\to 3} f(x) = 7$ and $g(3) = 4$, but $\lim_{x\to 3} f(x)g(x) \neq 28$, since $\lim_{x\to 3}(2x+1)/(x-3)$ does not exist.

91. True, by Property 2 of limits in Theorem 1.2.

92. True, by Properties 2 and 3 of limits in Theorem 1.2.

$$\lim_{x\to 3} g(x) = \lim_{x\to 3}(f(x) + g(x) + (-1)f(x)) = \lim_{x\to 3}(f(x) + g(x)) + (-1)\lim_{x\to 3} f(x) = 12 + (-1)7 = 5.$$

93. False. For example, define f as follows:

$$f(x) = \begin{cases} 2x+1 & \text{if } x \neq 2.99 \\ 1000 & \text{if } x = 2.99. \end{cases}$$

Then $f(2.9) = 2(2.9) + 1 = 6.8$, whereas $f(2.99) = 1000$.

94. False. For example, define f as follows:

$$f(x) = \begin{cases} 2x+1 & \text{if } x \neq 3.01 \\ -1000 & \text{if } x = 3.01. \end{cases}$$

Then $f(3.1) = 2(3.1) + 1 = 7.2$, whereas $f(3.01) = -1000$.

95. True. Suppose instead that $\lim_{x\to 3} g(x)$ does not exist but $\lim_{x\to 3}(f(x)g(x))$ did exist. Since $\lim_{x\to 3} f(x)$ exists and is not zero, then $\lim_{x\to 3}((f(x)g(x))/f(x))$ exists, by Property 4 of limits in Theorem 1.2. Furthermore, $f(x) \neq 0$ for all x in some interval about 3, so $(f(x)g(x))/f(x) = g(x)$ for all x in that interval. Thus $\lim_{x\to 3} g(x)$ exists. This contradicts our assumption that $\lim_{x\to 3} g(x)$ does not exist.

96. False. For some functions we need to pick smaller values of δ. For example, if $f(x) = x^{1/3} + 2$ and $c = 0$ and $L = 2$, then $f(x)$ is within 10^{-3} of 2 if $|x^{1/3}| < 10^{-3}$. This only happens if x is within $(10^{-3})^3 = 10^{-9}$ of 0. If $x = 10^{-3}$ then $x^{1/3} = (10^{-3})^{1/3} = 10^{-1}$, which is too large.

97. False. The definition of a limit guarantees that, for any positive ϵ, there is a δ. This statement, which guarantees an ϵ for a specific $\delta = 10^{-3}$, is not equivalent to $\lim_{x\to c} f(x) = L$. For example, consider a function with a vertical asymptote within 10^{-3} of 0, such as $c = 0$, $L = 0$, $f(x) = x/(x - 10^{-4})$.

98. True. This is equivalent to the definition of a limit.

99. False. Although x may be far from c, the value of $f(x)$ could be close to L. For example, suppose $f(x) = L$, the constant function.

100. False. The definition of the limit says that if x is within δ of c, then $f(x)$ is within ϵ of L, not the other way round.

101. (a) This statement follows: if we interchange the roles of f and g in the original statement, we get this statement.

(b) This statement is true, but it does not follow directly from the original statement, which says nothing about the case $g(a) = 0$.

(c) This follows, since if $g(a) \neq 0$ the original statement would imply f/g is continuous at $x = a$, but we are told it is not.

(d) This does not follow. Given that f is continuous at $x = a$ and $g(a) \neq 0$, then the original statement says g continuous implies f/g continuous, not the other way around. In fact, statement (d) is not true: if $f(x) = 0$ for all x, then g could be any discontinuous, non-zero function, and f/g would be zero, and therefore continuous. Thus the conditions of the statement would be satisfied, but not the conclusion.

Solutions for Chapter 1 Review

Exercises

1. The line of slope m through the point (x_0, y_0) has equation

$$y - y_0 = m(x - x_0),$$

so the line we want is

$$\begin{aligned} y - 0 &= 2(x - 5) \\ y &= 2x - 10. \end{aligned}$$

2. We want a function of the form $y = a(x-h)^2 + k$, with $a < 0$ because the parabola opens downward. Since (h, k) is the vertex, we must take $h = 2$, $k = 5$, but we can take any negative value of a. Figure 1.110 shows the graph with $a = -1$, namely $y = -(x-2)^2 + 5$.

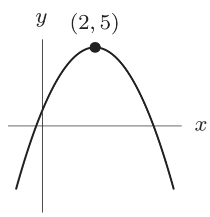


Figure 1.110: Graph of $y = -(x-2)^2 + 5$

3. A parabola with x-intercepts ± 1 has an equation of the form

$$y = k(x-1)(x+1).$$

Substituting the point $x = 0, y = 3$ gives

$$3 = k(-1)(1) \quad \text{so} \quad k = -3.$$

Thus, the equation we want is

$$\begin{aligned} y &= -3(x-1)(x+1) \\ y &= -3x^2 + 3. \end{aligned}$$

4. The equation of the whole circle is

$$x^2 + y^2 = (\sqrt{2})^2,$$

so the bottom half is

$$y = -\sqrt{2 - x^2}.$$

5. A circle with center (h, k) and radius r has equation $(x-h)^2 + (y-k)^2 = r^2$. Thus $h = -1$, $k = 2$, and $r = 3$, giving

$$(x+1)^2 + (y-2)^2 = 9.$$

Solving for y, and taking the positive square root gives the top half, so

$$\begin{aligned} (y-2)^2 &= 9 - (x+1)^2 \\ y &= 2 + \sqrt{9 - (x+1)^2}. \end{aligned}$$

See Figure 1.111.

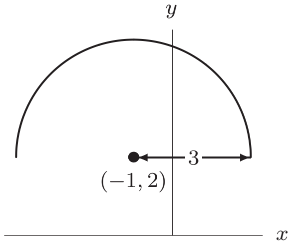


Figure 1.111: Graph of $y = 2 + \sqrt{9 - (x+1)^2}$

6. A cubic polynomial of the form $y = a(x-1)(x-5)(x-7)$ has the correct intercepts for any value of $a \neq 0$. Figure 1.112 shows the graph with $a = 1$, namely $y = (x-1)(x-5)(x-7)$.

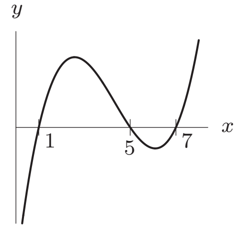


Figure 1.112: Graph of $y = (x-1)(x-5)(x-7)$

7. Since the vertical asymptote is $x = 2$, we have $b = -2$. The fact that the horizontal asymptote is $y = -5$ gives $a = -5$. So

$$y = \frac{-5x}{x-2}.$$

8. The amplitude of this function is 5, and its period is 2π, so $y = 5\cos x$.

9. See Figure 1.113.

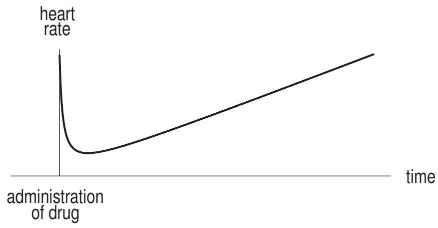


Figure 1.113

10. Factoring gives

$$g(x) = \frac{(2-x)(2+x)}{x(x+1)}.$$

The values of x which make $g(x)$ undefined are $x = 0$ and $x = -1$, when the denominator is 0. So the domain is all $x \neq 0, -1$. Solving $g(x) = 0$ means one of the numerator's factors is 0, so $x = \pm 2$.

11. (a) The domain of f is the set of values of x for which the function is defined. Since the function is defined by the graph and the graph goes from $x = 0$ to $x = 7$, the domain of f is $[0, 7]$.

(b) The range of f is the set of values of y attainable over the domain. Looking at the graph, we can see that y gets as high as 5 and as low as -2, so the range is $[-2, 5]$.

(c) Only at $x = 5$ does $f(x) = 0$. So 5 is the only zero of $f(x)$.

(d) Looking at the graph, we can see that $f(x)$ is decreasing on $(1, 7)$.

(e) The graph indicates that $f(x)$ is concave up at $x = 6$.

(f) The value $f(4)$ is the y-value that corresponds to $x = 4$. From the graph, we can see that $f(4)$ is approximately 1.

(g) This function is not invertible, since it fails the horizontal-line test. A horizontal line at $y = 3$ would cut the graph of $f(x)$ in two places, instead of the required one.

12. (a) $f(n) + g(n) = (3n^2 - 2) + (n + 1) = 3n^2 + n - 1$.
(b) $f(n)g(n) = (3n^2 - 2)(n + 1) = 3n^3 + 3n^2 - 2n - 2$.
(c) The domain of $f(n)/g(n)$ is defined everywhere where $g(n) \neq 0$, i.e. for all $n \neq -1$.
(d) $f(g(n)) = 3(n + 1)^2 - 2 = 3n^2 + 6n + 1$.
(e) $g(f(n)) = (3n^2 - 2) + 1 = 3n^2 - 1$.

13. (a) Since $m = f(A)$, we see that $f(100)$ represents the value of m when $A = 100$. Thus $f(100)$ is the minimum annual gross income needed (in thousands) to take out a 30-year mortgage loan of \$100,000 at an interest rate of 6%.
(b) Since $m = f(A)$, we have $A = f^{-1}(m)$. We see that $f^{-1}(75)$ represents the value of A when $m = 75$, or the size of a mortgage loan that could be obtained on an income of \$75,000.

14. Taking logs of both sides yields $t \log 5 = \log 7$, so $t = \dfrac{\log 7}{\log 5} = 1.209$.

15. $t = \dfrac{\log 2}{\log 1.02} \approx 35.003$.

16. Collecting similar factors yields $\left(\frac{3}{2}\right)^t = \frac{5}{7}$, so

$$t = \frac{\log\left(\frac{5}{7}\right)}{\log\left(\frac{3}{2}\right)} = -0.830.$$

17. Collecting similar factors yields $\left(\frac{1.04}{1.03}\right)^t = \frac{12.01}{5.02}$. Solving for t yields

$$t = \frac{\log\left(\frac{12.01}{5.02}\right)}{\log\left(\frac{1.04}{1.03}\right)} = 90.283.$$

18. We want $2^t = e^{kt}$ so $2 = e^k$ and $k = \ln 2 = 0.693$. Thus $P = P_0 e^{0.693t}$.

19. We want $0.2^t = e^{kt}$ so $0.2 = e^k$ and $k = \ln 0.2 = -1.6094$. Thus $P = 5.23e^{-1.6094t}$.

20. $f(x) = \ln x, \quad g(x) = x^3$. (Another possibility: $f(x) = 3x, \quad g(x) = \ln x$.)

21. $f(x) = x^3, \quad g(x) = \ln x$.

22. The amplitude is 5. The period is 6π. See Figure 1.114.

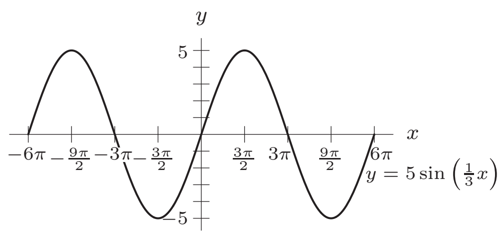


Figure 1.114

23. The amplitude is 2. The period is $2\pi/5$. See Figure 1.115.

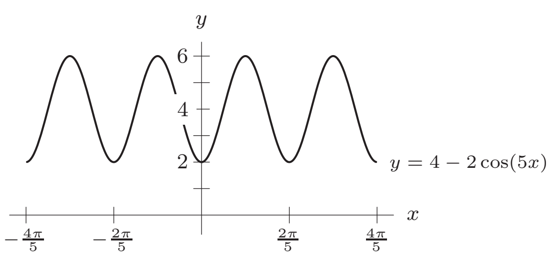
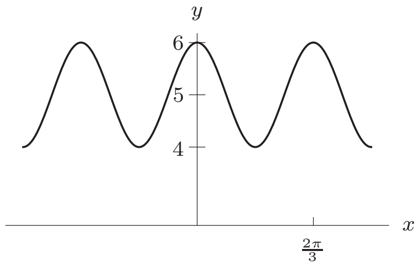


Figure 1.115

24. (a) We determine the amplitude of y by looking at the coefficient of the cosine term. Here, the coefficient is 1, so the amplitude of y is 1. Note that the constant term does not affect the amplitude.
(b) We know that the cosine function $\cos x$ repeats itself at $x = 2\pi$, so the function $\cos(3x)$ must repeat itself when $3x = 2\pi$, or at $x = 2\pi/3$. So the period of y is $2\pi/3$. Here as well the constant term has no effect.
(c) The graph of y is shown in the figure below.



25. (a) Since $f(x)$ is an odd polynomial with a positive leading coefficient, it follows that $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$.
(b) Since $f(x)$ is an even polynomial with negative leading coefficient, it follows that $f(x) \to -\infty$ as $x \to \pm\infty$.
(c) As $x \to \pm\infty$, $x^4 \to \infty$, so $x^{-4} = 1/x^4 \to 0$.
(d) As $x \to \pm\infty$, the lower-degree terms of $f(x)$ become insignificant, and $f(x)$ becomes approximated by the highest degree terms in its numerator and denominator. So as $x \to \pm\infty$, $f(x) \to 6$.

26. Exponential growth dominates power growth as $x \to \infty$, so $10 \cdot 2^x$ is larger.

27. As $x \to \infty$, $0.25x^{1/2}$ is larger than $25{,}000x^{-3}$.

28. This is a line with slope $-3/7$ and y-intercept 3, so a possible formula is

$$y = -\frac{3}{7}x + 3.$$

29. Starting with the general exponential equation $y = Ae^{kx}$, we first find that for $(0, 1)$ to be on the graph, we must have $A = 1$. Then to make $(3, 4)$ lie on the graph, we require

$$\begin{aligned} 4 &= e^{3k} \\ \ln 4 &= 3k \\ k &= \frac{\ln 4}{3} \approx 0.4621. \end{aligned}$$

Thus the equation is

$$y = e^{0.4621x}.$$

Alternatively, we can use the form $y = a^x$, in which case we find $y = (1.5874)^x$.

30. This looks like an exponential function. The y-intercept is 3 and we use the form $y = 3e^{kt}$. We substitute the point $(5, 9)$ to solve for k:

$$\begin{aligned} 9 &= 3e^{k5} \\ 3 &= e^{5k} \\ \ln 3 &= 5k \\ k &= 0.2197. \end{aligned}$$

A possible formula is

$$y = 3e^{0.2197t}.$$

Alternatively, we can use the form $y = 3a^t$, in which case we find $y = 3(1.2457)^t$.

31. $y = -kx(x+5) = -k(x^2+5x)$, where $k > 0$ is any constant.

32. Since this function has a y-intercept at $(0, 2)$, we expect it to have the form $y = 2e^{kx}$. Again, we find k by forcing the other point to lie on the graph:

$$\begin{aligned} 1 &= 2e^{2k} \\ \frac{1}{2} &= e^{2k} \\ \ln\left(\frac{1}{2}\right) &= 2k \\ k &= \frac{\ln(\frac{1}{2})}{2} \approx -0.34657. \end{aligned}$$

This value is negative, which makes sense since the graph shows exponential decay. The final equation, then, is

$$y = 2e^{-0.34657x}.$$

Alternatively, we can use the form $y = 2a^x$, in which case we find $y = 2(0.707)^x$.

33. $z = 1 - \cos\theta$

34. $y = k(x+2)(x+1)(x-1) = k(x^3+2x^2-x-2)$, where $k > 0$ is any constant.

35. $x = ky(y-4) = k(y^2-4y)$, where $k > 0$ is any constant.

36. $y = 5\sin\left(\frac{\pi t}{20}\right)$

37. This looks like a fourth degree polynomial with roots at -5 and -1 and a double root at 3. The leading coefficient is negative, and so a possible formula is

$$y = -(x+5)(x+1)(x-3)^2.$$

38. This looks like a rational function. There are vertical asymptotes at $x = -2$ and $x = 2$ and so one possibility for the denominator is $x^2 - 4$. There is a horizontal asymptote at $y = 3$ and so the numerator might be $3x^2$. In addition, $y(0) = 0$ which is the case with the numerator of $3x^2$. A possible formula is

$$y = \frac{3x^2}{x^2-4}.$$

39. There are many solutions for a graph like this one. The simplest is $y = 1 - e^{-x}$, which gives the graph of $y = e^x$, flipped over the x-axis and moved up by 1. The resulting graph passes through the origin and approaches $y = 1$ as an upper bound, the two features of the given graph.

40. The graph is a sine curve which has been shifted up by 2, so $f(x) = (\sin x) + 2$.

41. This graph has period 5, amplitude 1 and no vertical shift or horizontal shift from $\sin x$, so it is given by

$$f(x) = \sin\left(\frac{2\pi}{5}x\right).$$

42. Since the denominator, $x^2 + 1$, is continuous and never zero, $g(x)$ is continuous on $[-1, 1]$.

43. Since

$$h(x) = \frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)},$$

we see that $h(x)$ is not defined at $x = -1$ or at $x = 1$, so $h(x)$ is not continuous on $[-1, 1]$.

44. (a) $\lim_{x\to 0} f(x) = 1$.
(b) $\lim_{x\to 1} f(x)$ does not exist.
(c) $\lim_{x\to 2} f(x) = 1$.
(d) $\lim_{x\to 3^-} f(x) = 0$.

45. $f(x) = \dfrac{x^3|2x-6|}{x-3} = \begin{cases} \dfrac{x^3(2x-6)}{x-3} = 2x^3, & x > 3 \\ \dfrac{x^3(-2x+6)}{x-3} = -2x^3, & x < 3 \end{cases}$

Figure 1.116 confirms that $\lim_{x\to 3^+} f(x) = 54$ while $\lim_{x\to 3^-} f(x) = -54$; thus $\lim_{x\to 3} f(x)$ does not exist.

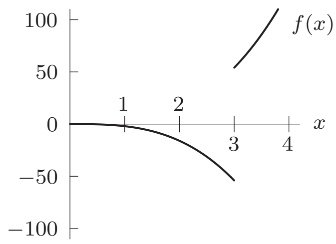


Figure 1.116

46. $f(x) = \begin{cases} e^x & -1 < x < 0 \\ 1 & x = 0 \\ \cos x & 0 < x < 1 \end{cases}$

Figure 1.117 confirms that $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^x = e^0 = 1$, and that $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \cos x = \cos 0 = 1$, so $\lim_{x\to 0} f(x) = 1$.

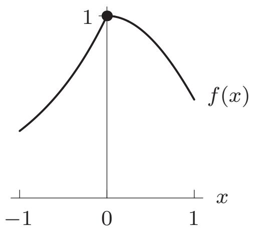


Figure 1.117

Problems

47. (a) More fertilizer increases the yield until about 40 lbs.; then it is too much and ruins crops, lowering yield.
(b) The vertical intercept is at $Y = 200$. If there is no fertilizer, then the yield is 200 bushels.
(c) The horizontal intercept is at $a = 80$. If you use 80 lbs. of fertilizer, then you will grow no apples at all.
(d) The range is the set of values of Y attainable over the domain $0 \le a \le 80$. Looking at the graph, we can see that Y goes as high as 550 and as low as 0. So the range is $0 \le Y \le 550$.
(e) Looking at the graph, we can see that Y is decreasing at $a = 60$.
(f) Looking at the graph, we can see that Y is concave down everywhere, so it is certainly concave down at $a = 40$.

48. (a) Given the two points $(0, 32)$ and $(100, 212)$, and assuming the graph in Figure 1.118 is a line,

$$\text{Slope} = \frac{212-32}{100} = \frac{180}{100} = 1.8.$$

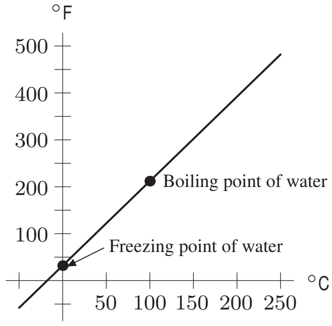


Figure 1.118

(b) The °F-intercept is $(0, 32)$, so

$$^\circ\text{F} = 1.8(^\circ\text{C}) + 32.$$

(c) If the temperature is 20°Celsius, then

$$^\circ\text{F} = 1.8(20) + 32 = 68^\circ\text{F}.$$

(d) If °F = °C, then

$$\begin{aligned} ^\circ\text{C} &= 1.8^\circ\text{C} + 32 \\ -32 &= 0.8^\circ\text{C} \\ ^\circ\text{C} &= -40^\circ = {}^\circ\text{F}. \end{aligned}$$

49. (a) We have the following functions.

(i) Since a change in p of \$5 results in a decrease in q of 2, the slope of $q = D(p)$ is $-2/5$ items per dollar. So

$$q = b - \frac{2}{5}p.$$

Now we know that when $p = 550$ we have $q = 100$, so

$$\begin{aligned} 100 &= b - \frac{2}{5} \cdot 550 \\ 100 &= b - 220 \\ b &= 320. \end{aligned}$$

Thus a formula is

$$q = 320 - \frac{2}{5}p.$$

(ii) We can solve $q = 320 - \frac{2}{5}p$ for p in terms of q:

$$\begin{aligned} 5q &= 1600 - 2p \\ 2p &= 1600 - 5q \\ p &= 800 - \frac{5}{2}q. \end{aligned}$$

The slope of this function is $-5/2$ dollars per item, as we would expect.

(b) A graph of $p = 800 - \dfrac{5}{2}q$ is given in Figure 1.119.

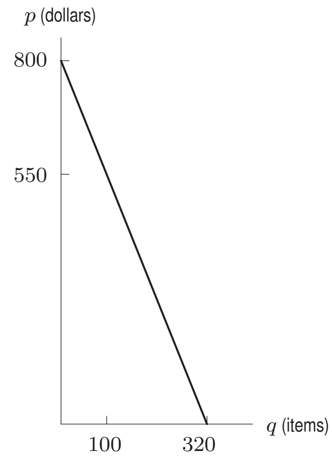


Figure 1.119

50. See Figure 1.120.

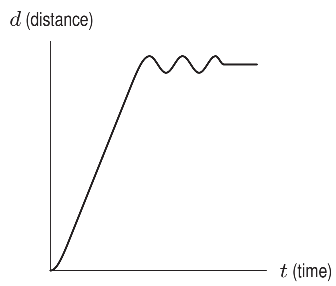


Figure 1.120

51. (a) (i) If the atoms are moved farther apart, then $r > a$ so, from the graph, F is negative, indicating an attractive force, which pulls the atoms back together.

(ii) If the atoms are moved closer together, then $r < a$ so, from the graph, F is positive, indicating an attractive force, which pushes the atoms apart again.

(b) At $r = a$, the force is zero. The answer to part (a)(i) tells us that if the atoms are pulled apart slightly, so $r > a$, the force tends to pull them back together; the answer to part (a)(ii) tells us that if the atoms are pushed together, so $r < a$, the force tends to push them back apart. Thus, $r = a$ is a stable equilibrium.

52. If the pressure at sea level is P_0, the pressure P at altitude h is given by

$$P = P_0\left(1 - \frac{0.4}{100}\right)^{h/100},$$

since we want the pressure to be multiplied by a factor of $(1 - 0.4/100) = 0.996$ for each 100 feet we go up to make it decrease by 0.4% over that interval. At Mexico City $h = 7340$, so the pressure is

$$P = P_0(0.996)^{7340/100} \approx 0.745P_0.$$

So the pressure is reduced from P_0 to approximately $0.745P_0$, a decrease of 25.5%.

53. Assuming the population of Ukraine is declining exponentially, we have population $P(t) = 45.7e^{kt}$ at time t years after 2009. Using the 2010 population, we have

$$45.42 = 45.7e^{-k\cdot 1}$$
$$k = -\ln\left(\frac{45.42}{45.7}\right) = 0.0061.$$

We want to find the time t at which

$$45 = 45.7e^{-0.0061t}$$
$$t = -\frac{\ln(45/45.7)}{0.0061} = 2.53 \text{ years.}$$

This model predicts the population to go below 45 million 2.53 years after 2009, in the year 2011.

54. (a) We compound the daily inflation rate 30 times to get the desired monthly rate r:

$$\left(1+\frac{r}{100}\right)^1 = \left(1+\frac{0.67}{100}\right)^{30} = 1.2218.$$

Solving for r, we get $r = 22.18$, so the inflation rate for April was 22.18%.

(b) We compound the daily inflation rate 365 times to get a yearly rate R for 2006:

$$\left(1+\frac{R}{100}\right)^1 = \left(1+\frac{0.67}{100}\right)^{365} = 11.4426.$$

Solving for R, we get $R = 1044.26$, so the yearly rate was 1044.26% during 2006. We could have obtained approximately the same result by compounding the monthly rate 12 times. Computing the annual rate from the monthly gives a lower result, because 12 months of 30 days each is only 360 days.

55. (a) The US consumption of hydroelectric power increased by at least 10% in 2009 and decreased by at least 10% in 2006 and in 2007, relative to each corresponding previous year. In 2009 consumption increased by 11% over consumption in 2008. In 2006 consumption decreased by 10% over consumption in 2005, and in 2007 consumption decreased by about 45% over consumption in 2006.

(b) False. In 2009 hydroelectric power consumption increased only by 11% over consumption in 2008.

(c) True. From 2006 to 2007 consumption decreased by 45.4%, which means $x(1-0.454)$ units of hydroelectric power were consumed in 2007 if x had been consumed in 2006. Similarly,

$$(x(1-0.454))(1+0.051)$$

units of hydroelectric power were consumed in 2008 if x had been consumed in 2006, and

$$(x(1-0.454)(1+0.051))(1+0.11)$$

units of hydroelectric power were consumed in 2009 if x had been consumed in 2006. Since

$$x(1-0.454)(1+0.051)(1+0.11) = x(0.637) = x(1-0.363),$$

the percent growth in hydroelectric power consumption was -36.3%, in 2009 relative to consumption in 2006. This amounts to about 36% decrease in hydroelectric power consumption from 2006 to 2009.

56. (a) For each 2.2 pounds of weight the object has, it has 1 kilogram of mass, so the conversion formula is

$$k = f(p) = \frac{1}{2.2}p.$$

(b) The inverse function is

$$p = 2.2k,$$

and it gives the weight of an object in pounds as a function of its mass in kilograms.

57. Since $f(x)$ is a parabola that opens upward, we have $f(x) = ax^2 + bx + c$ with $a > 0$. Since $g(x)$ is a line with negative slope, $g(x) = b + mx$, with slope $m < 0$. Therefore

$$g(f(x)) = b + m(ax^2 + bx + c) = max^2 + mbx + mc + b.$$

The coefficient of x^2 is ma, which is negative. Thus, the graph is a parabola opening downward.

58. (a) is $g(x)$ since it is linear. (b) is $f(x)$ since it has decreasing slope; the slope starts out about 1 and then decreases to about $\frac{1}{10}$. (c) is $h(x)$ since it has increasing slope; the slope starts out about $\frac{1}{10}$ and then increases to about 1.

59. Given the doubling time of 2 hours, $200 = 100e^{k(2)}$, we can solve for the growth rate k using the equation:

$$2P_0 = P_0e^{2k}$$
$$\ln 2 = 2k$$
$$k = \frac{\ln 2}{2}.$$

Using the growth rate, we wish to solve for the time t in the formula

$$P = 100e^{\frac{\ln 2}{2}t}$$

where $P = 3{,}200$, so

$$\begin{aligned} 3{,}200 &= 100e^{\frac{\ln 2}{2}t} \\ t &= 10 \text{ hours}. \end{aligned}$$

60. (a) The y-intercept of $f(x) = a\ln(x+2)$ is $f(0) = a\ln 2$. Since $\ln 2$ is positive, increasing a increases the y-intercept.
(b) The x-intercept of $f(x) = a\ln(x+2)$ is where $f(x) = 0$. Since this occurs where $x+2 = 1$, so $x = -1$, increasing a does not affect the x-intercept.

61. Since the factor by which the prices have increased after time t is given by $(1.05)^t$, the time after which the prices have doubled solves

$$\begin{aligned} 2 &= (1.05)^t \\ \log 2 &= \log(1.05^t) = t\log(1.05) \\ t &= \frac{\log 2}{\log 1.05} \approx 14.21 \text{ years}. \end{aligned}$$

62. Using the exponential decay equation $P = P_0e^{-kt}$, we can solve for the substance's decay constant k:

$$\begin{aligned} (P_0 - 0.3P_0) &= P_0e^{-20k} \\ k &= \frac{\ln(0.7)}{-20}. \end{aligned}$$

Knowing this decay constant, we can solve for the half-life t using the formula

$$\begin{aligned} 0.5P_0 &= P_0e^{\ln(0.7)t/20} \\ t &= \frac{20\ln(0.5)}{\ln(0.7)} \approx 38.87 \text{ hours}. \end{aligned}$$

63. (a) We know the decay follows the equation

$$P = P_0e^{-kt},$$

and that 10% of the pollution is removed after 5 hours (meaning that 90% is left). Therefore,

$$\begin{aligned} 0.90P_0 &= P_0e^{-5k} \\ k &= -\frac{1}{5}\ln(0.90). \end{aligned}$$

Thus, after 10 hours:

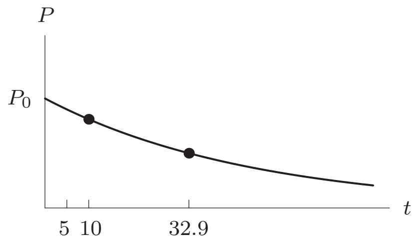
$$\begin{aligned} P &= P_0e^{-10((-0.2)\ln 0.90)} \\ P &= P_0(0.9)^2 = 0.81P_0 \end{aligned}$$

so 81% of the original amount is left.

(b) We want to solve for the time when $P = 0.50P_0$:

$$\begin{aligned} 0.50P_0 &= P_0e^{t((0.2)\ln 0.90)} \\ 0.50 &= e^{\ln(0.90^{0.2t})} \\ 0.50 &= 0.90^{0.2t} \\ t &= \frac{5\ln(0.50)}{\ln(0.90)} \approx 32.9 \text{ hours}. \end{aligned}$$

(c)



(d) When highly polluted air is filtered, there is more pollutant per liter of air to remove. If a fixed amount of air is cleaned every day, there is a higher amount of pollutant removed earlier in the process.

64. Since the amount of strontium-90 remaining halves every 29 years, we can solve for the decay constant;

$$0.5P_0 = P_0 e^{-29k}$$
$$k = \frac{\ln(1/2)}{-29}.$$

Knowing this, we can look for the time t in which $P = 0.10P_0$, or

$$0.10P_0 = P_0 e^{\ln(0.5)t/29}$$
$$t = \frac{29\ln(0.10)}{\ln(0.5)} = 96.336 \text{ years.}$$

65. One hour.

66. (a) V_0 represents the maximum voltage.
(b) The period is $2\pi/(120\pi) = 1/60$ second.
(c) Since each oscillation takes $1/60$ second, in 1 second there are 60 complete oscillations.

67. The US voltage has a maximum value of 156 volts and has a period of $1/60$ of a second, so it executes 60 cycles a second.
The European voltage has a higher maximum of 339 volts, and a slightly longer period of $1/50$ seconds, so it oscillates at 50 cycles per second.

68. (a) The amplitude of the sine curve is $|A|$. Thus, increasing $|A|$ stretches the curve vertically. See Figure 1.121.
(b) The period of the wave is $2\pi/|B|$. Thus, increasing $|B|$ makes the curve oscillate more rapidly—in other words, the function executes one complete oscillation in a smaller interval. See Figure 1.122.

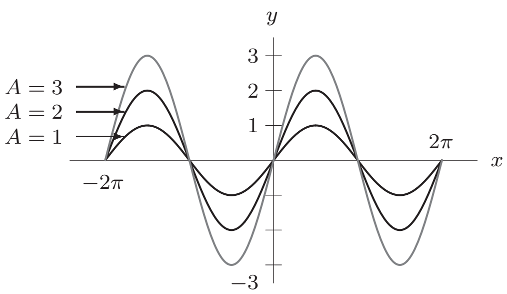


Figure 1.121

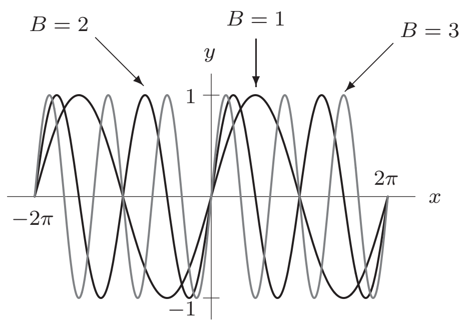


Figure 1.122

69. (a) (i) The water that has flowed out of the pipe in 1 second is a cylinder of radius r and length 3 cm. Its volume is

$$V = \pi r^2(3) = 3\pi r^2.$$

(ii) If the rate of flow is k cm/sec instead of 3 cm/sec, the volume is given by

$$V = \pi r^2(k) = \pi r^2 k.$$

(b) (i) The graph of V as a function of r is a quadratic. See Figure 1.123.

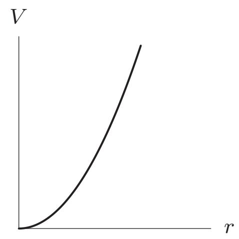


Figure 1.123

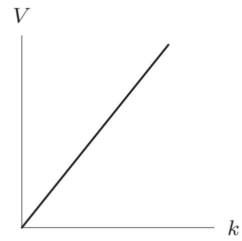


Figure 1.124

(ii) The graph of V as a function of k is a line. See Figure 1.124.

70. Looking at g, we see that the ratio of the values is:

$$\frac{3.12}{3.74} \approx \frac{3.74}{4.49} \approx \frac{4.49}{5.39} \approx \frac{5.39}{6.47} \approx \frac{6.47}{7.76} \approx 0.83.$$

Thus g is an exponential function, and so f and k are the power functions. Each is of the form ax^2 or ax^3, and since $k(1.0) = 9.01$ we see that for k, the constant coefficient is 9.01. Trial and error gives

$$k(x) = 9.01x^2,$$

since $k(2.2) = 43.61 \approx 9.01(4.84) = 9.01(2.2)^2$. Thus $f(x) = ax^3$ and we find a by noting that $f(9) = 7.29 = a(9^3)$ so

$$a = \frac{7.29}{9^3} = 0.01$$

and $f(x) = 0.01x^3$.

71. (a) See Figure 1.125.

(b) The graph is made of straight line segments, rising from the x-axis at the origin to height a at $x = 1$, b at $x = 2$, and c at $x = 3$ and then returning to the x-axis at $x = 4$. See Figure 1.126.

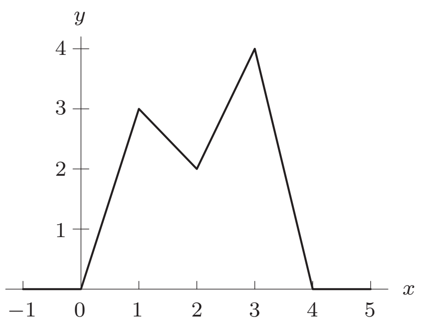


Figure 1.125

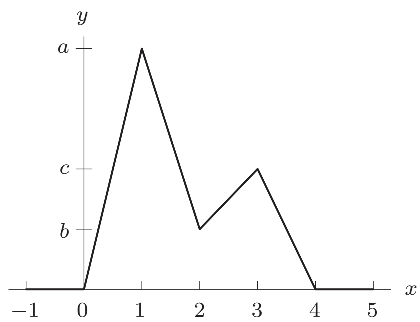


Figure 1.126

72. (a) Reading the graph of θ against t shows that $\theta \approx 5.2$ when $t = 1.5$. Since the coordinates of P are $x = 5\cos\theta$, $y = 5\sin\theta$, when $t = 1.5$ the coordinates are

$$(x, y) \approx (5\cos 5.2, 5\sin 5.2) = (2.3, -4.4).$$

(b) As t increases from 0 to 5, the angle θ increases from 0 to about 6.3 and then decreases to 0 again. Since $6.3 \approx 2\pi$, this means that P starts on the x-axis at the point $(5, 0)$, moves counterclockwise the whole way around the circle (at which time $\theta \approx 2\pi$), and then moves back clockwise to its starting point.

73. (a) III
(b) IV
(c) I
(d) II

74. The functions $y(x) = \sin x$ and $z_k(x) = ke^{-x}$ for $k = 1, 2, 4, 6, 8, 10$ are shown in Figure 1.127. The values of $f(k)$ for $k = 1, 2, 4, 6, 8, 10$ are given in Table 1.16. These values can be obtained using either tracing or a numerical root finder on a calculator or computer.

From Figure 1.127 it is clear that the smallest solution of $\sin x = ke^{-x}$ for $k = 1, 2, 4, 6$ occurs on the first period of the sine curve. For small changes in k, there are correspondingly small changes in the intersection point. For $k = 8$ and $k = 10$, the solution jumps to the second period because $\sin x < 0$ between π and 2π, but ke^{-x} is uniformly positive. Somewhere in the interval $6 \le k \le 8$, $f(k)$ has a discontinuity.

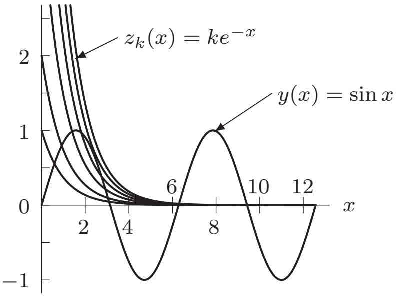


Figure 1.127

Table 1.16

k	$f(k)$
1	0.588
2	0.921
4	1.401
6	1.824
8	6.298
10	6.302

75. By tracing on a calculator or solving equations, we find the following values of δ:
For $\epsilon = 0.1$, $\delta \le 0.1$
For $\epsilon = 0.05$, $\delta \le 0.05$.
For $\epsilon = 0.0007$, $\delta \le 0.00007$.

76. By tracing on a calculator or solving equations, we find the following values of δ:
For $\epsilon = 0.1$, $\delta \le 0.45$.
For $\epsilon = 0.001$, $\delta \le 0.0447$.
For $\epsilon = 0.00001$, $\delta \le 0.00447$.

77. For any values of k, the function is continuous on any interval that does not contain $x = 2$.

Since $5x^3 - 10x^2 = 5x^2(x - 2)$, we can cancel $(x - 2)$ provided $x \ne 2$, giving

$$f(x) = \frac{5x^3 - 10x^2}{x - 2} = 5x^2 \qquad x \ne 2.$$

Thus, if we pick $k = 5(2)^2 = 20$, the function is continuous.

78. At $x = 0$, the curve $y = k \cos x$ has $y = k \cos 0 = k$. At $x = 0$, the curve $y = e^x - k$ has $y = e^0 - k = 1 - k$. If $j(x)$ is continuous, we need

$$k = 1 - k, \quad \text{so} \quad k = \frac{1}{2}.$$

CAS Challenge Problems

79. (a) A CAS gives $f(x) = (x - a)(x + a)(x + b)(x - c)$.

(b) The graph of $f(x)$ crosses the x-axis at $x = a$, $x = -a$, $x = -b$, $x = c$; it crosses the y-axis at a^2bc. Since the coefficient of x^4 (namely 1) is positive, the graph of f looks like that shown in Figure 1.128.

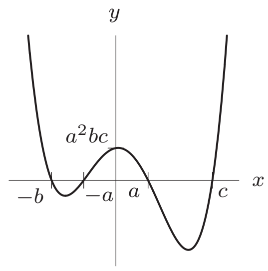


Figure 1.128: Graph of $f(x) = (x-a)(x+a)(x+b)(x-c)$

80. (a) A CAS gives $f(x) = -(x-1)^2(x-3)^3$.

(b) For large $|x|$, the graph of $f(x)$ looks like the graph of $y = -x^5$, so $f(x) \to \infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to \infty$. The answer to part (a) shows that f has a double root at $x = 1$, so near $x = 1$, the graph of f looks like a parabola touching the x-axis at $x = 1$. Similarly, f has a triple root at $x = 3$. Near $x = 3$, the graph of f looks like the graph of $y = x^3$, flipped over the x-axis and shifted to the right by 3, so that the "seat" is at $x = 3$. See Figure 1.129.

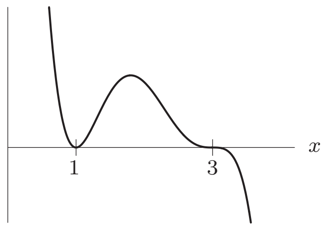


Figure 1.129: Graph of $f(x) = -(x-1)^2(x-3)^3$

81. (a) As $x \to \infty$, the term e^{6x} dominates and tends to ∞. Thus, $f(x) \to \infty$ as $x \to \infty$.
As $x \to -\infty$, the terms of the form e^{kx}, where $k = 6, 5, 4, 3, 2, 1$, all tend to zero. Thus, $f(x) \to 16$ as $x \to -\infty$.

(b) A CAS gives

$$f(x) = (e^x + 1)(e^{2x} - 2)(e^x - 2)(e^{2x} + 2e^x + 4).$$

Since e^x is always positive, the factors $(e^x + 1)$ and $(e^{2x} + 2e^x + 4)$ are never zero. The other factors each lead to a zero, so there are two zeros.

(c) The zeros are given by

$$e^{2x} = 2 \quad \text{so} \quad x = \frac{\ln 2}{2}$$
$$e^x = 2 \quad \text{so} \quad x = \ln 2.$$

Thus, one zero is twice the size of the other.

82. (a) Since $f(x) = x^2 - x$,

$$f(f(x)) = (f(x))^2 - f(x) = (x^2 - x)^2 - (x^2 - x) = x - 2\,x^3 + x^4.$$

Using the CAS to define the function $f(x)$, and then asking it to expand $f(f(f(x)))$, we get

$$f(f(f(x))) = -x + x^2 + 2\,x^3 - 5\,x^4 + 2\,x^5 + 4\,x^6 - 4\,x^7 + x^8.$$

(b) The degree of $f(f(x))$ (that is, f composed with itself 2 times) is $4 = 2^2$. The degree of $f(f(f(x)))$ (that is, f composed with itself 3 times), is $8 = 2^3$. Each time you substitute f into itself, the degree is multiplied by 2, because you are substituting in a degree 2 polynomial. So we expect the degree of $f(f(f(f(f(f(x))))))$ (that is, f composed with itself 6 times) to be $64 = 2^6$.

83. (a) A CAS or division gives

$$f(x) = \frac{x^3 - 30}{x - 3} = x^2 + 3x + 9 - \frac{3}{x - 3},$$

so $p(x) = x^2 + 3x + 9$, and $r(x) = -3$, and $q(x) = x - 3$.

(b) The vertical asymptote is $x = 3$. Near $x = 3$, the values of $p(x)$ are much smaller than the values of $r(x)/q(x)$. Thus

$$f(x) \approx \frac{-3}{x - 3} \qquad \text{for } x \text{ near } 3.$$

(c) For large x, the values of $p(x)$ are much larger than the value of $r(x)/q(x)$. Thus

$$f(x) \approx x^2 + 3x + 9 \qquad \text{as } x \to \infty, x \to -\infty.$$

(d) Figure 1.130 shows $f(x)$ and $y = -3/(x - 3)$ for x near 3. Figure 1.131 shows $f(x)$ and $y = x^2 + 3x + 9$ for $-20 \le x \le 20$. Note that in each case the graphs of f and the approximating function are close.

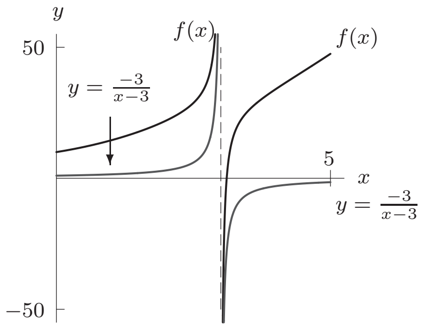


Figure 1.130: Close-up view of $f(x)$ and $y=-3/(x-3)$

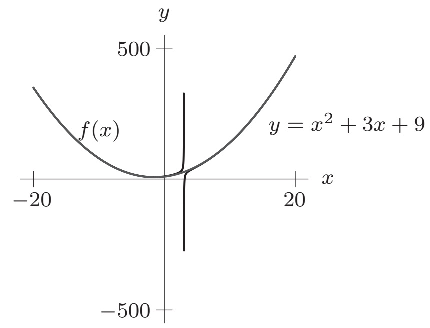


Figure 1.131: Far-away view of $f(x)$ and $y=x^2+3x+9$

84. Using the trigonometric expansion capabilities of your CAS, you get something like

$$\sin(5x) = 5\cos^4(x)\,\sin(x) - 10\cos^2(x)\sin^3(x) + \sin^5(x).$$

Answers may vary. To get rid of the powers of cosine, use the identity $\cos^2(x) = 1 - \sin^2(x)$. This gives

$$\sin(5x) = 5\,\sin(x)\left(1-\sin^2(x)\right)^2 - 10\sin^3(x)\,\left(1-\sin^2(x)\right) + \sin^5(x).$$

Finally, using the CAS to simplify,

$$\sin(5x) = 5\,\sin(x) - 20\sin^3(x) + 16\sin^5(x).$$

85. Using the trigonometric expansion capabilities of your computer algebra system, you get something like

$$\cos(4x) = \cos^4(x) - 6\cos^2(x)\sin^2(x) + \sin^4(x).$$

Answers may vary.

(a) To get rid of the powers of cosine, use the identity $\cos^2(x) = 1 - \sin^2(x)$. This gives

$$\cos(4x) = \cos^4(x) - 6\cos^2(x)\,\left(1-\cos^2(x)\right) + \left(1-\cos^2(x)\right)^2.$$

Finally, using the CAS to simplify,

$$\cos(4x) = 1 - 8\cos^2(x) + 8\cos^4(x).$$

(b) This time we use $\sin^2(x) = 1-\cos^2(x)$ to get rid of powers of sine. We get

$$\cos(4x) = \left(1-\sin^2(x)\right)^2 - 6\sin^2(x)\,\left(1-\sin^2(x)\right) + \sin^4(x) = 1 - 8\sin^2(x) + 8\sin^4(x).$$

PROJECTS FOR CHAPTER ONE

1. Notice that whenever x increases by 0.5, $f(x)$ increases by 1, indicating that $f(x)$ is linear. By inspection, we see that $f(x) = 2x$.

Similarly, $g(x)$ decreases by 1 each time x increases by 0.5. We know, therefore, that $g(x)$ is a linear function with slope $\frac{-1}{0.5} = -2$. The y-intercept is 10, so $g(x) = 10 - 2x$.

$h(x)$ is an even function which is always positive. Comparing the values of x and $h(x)$, it appears that $h(x) = x^2$.

$F(x)$ is an odd function that seems to vary between -1 and 1. We guess that $F(x) = \sin x$ and check with a calculator.

$G(x)$ is also an odd function that varies between -1 and 1. Notice that $G(x) = F(2x)$, and thus $G(x) = \sin 2x$.

Notice also that $H(x)$ is exactly 2 more than $F(x)$ for all x, so $H(x) = 2 + \sin x$.

2. (a) Begin by finding a table of correspondences between the mathematicians' and meteorologists' angles.

θ_{met} (in degrees)	0	45	90	135	180	225	270	315
θ_{math} (in degrees)	270	225	180	135	90	45	0	315

The table is linear for $0 \le \theta_{\text{met}} \le 270$, with θ_{math} decreasing by 45 every time θ_{met} increases by 45, giving slope $\Delta\theta_{\text{met}}/\Delta\theta_{\text{math}} = 45/(-45) = -1$.

The interval $270 < \theta_{\text{met}} < 360$ needs a closer look. We have the following more detailed table for that interval:

θ_{met}	280	290	300	310	320	330	340	350
θ_{math}	350	340	330	320	310	300	290	280

Again the table is linear, this time with θ_{math} decreasing by 10 every time θ_{met} increases by 10, again giving slope -1. The graph of θ_{math} against θ_{met} contains two straight line sections, both of slope -1. See Figure 1.132.

(b) See Figure 1.132.

$$\theta_{\text{math}} = \begin{cases} 270 - \theta_{\text{met}} & \text{if } 0 \le \theta_{\text{met}} \le 270 \\ 630 - \theta_{\text{met}} & \text{if } 270 < \theta_{\text{met}} < 360 \end{cases}.$$

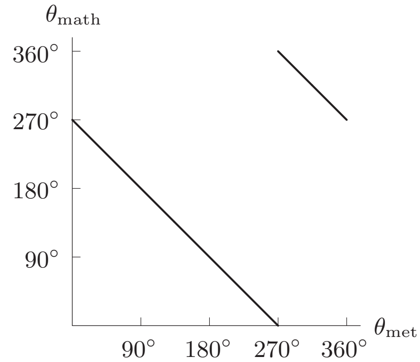


Figure 1.132