CHAPTER 1 – 9th Edition

1.1. If **A** represents a vector two units in length directed due west, **B** represents a vector three units in length directed due north, and $\mathbf{A} + \mathbf{B} = \mathbf{C} - \mathbf{D}$ and $2\mathbf{B} - \mathbf{A} = \mathbf{C} + \mathbf{D}$, find the magnitudes and directions of **C** and **D**. Take north as the positive y direction:

With north as positive y, west will be $-x$. We may therefore set up:

$$\mathbf{C} + \mathbf{D} = 2\mathbf{B} - \mathbf{A} = 6\mathbf{a}_y + 2\mathbf{a}_x \text{ and}$$
$$\mathbf{C} - \mathbf{D} = \mathbf{A} + \mathbf{B} = -2\mathbf{a}_x + 3\mathbf{a}_y$$

Add the equations to find $\underline{\mathbf{C} = 4.5\mathbf{a}_y \text{ (north)}}$, and then $\underline{\mathbf{D} = 2\mathbf{a}_x + 1.5\mathbf{a}_y \text{ (east of northeast)}}$.

1.2. Vector **A** extends from the origin to (1,2,3) and vector **B** from the origin to (2,3,-2).

a) Find the unit vector in the direction of $(\mathbf{A} - \mathbf{B})$: First

$$\mathbf{A} - \mathbf{B} = (\mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z) - (2\mathbf{a}_x + 3\mathbf{a}_y - 2\mathbf{a}_z) = (-\mathbf{a}_x - \mathbf{a}_y + 5\mathbf{a}_z)$$

whose magnitude is $|\mathbf{A} - \mathbf{B}| = \left[(-\mathbf{a}_x - \mathbf{a}_y + 5\mathbf{a}_z) \cdot (-\mathbf{a}_x - \mathbf{a}_y + 5\mathbf{a}_z)\right]^{1/2} = \sqrt{1+1+25} = 3\sqrt{3} = 5.20$. The unit vector is therefore

$$\mathbf{a}_{AB} = \underline{(-\mathbf{a}_x - \mathbf{a}_y + 5\mathbf{a}_z)/5.20}$$

b) find the unit vector in the direction of the line extending from the origin to the midpoint of the line joining the ends of **A** and **B**:

The midpoint is located at

$$P_{mp} = [1 + (2-1)/2,\ 2 + (3-2)/2,\ 3 + (-2-3)/2)] = (1.5, 2.5, 0.5)$$

The unit vector is then

$$\mathbf{a}_{mp} = \frac{(1.5\mathbf{a}_x + 2.5\mathbf{a}_y + 0.5\mathbf{a}_z)}{\sqrt{(1.5)^2 + (2.5)^2 + (0.5)^2}} = \underline{(1.5\mathbf{a}_x + 2.5\mathbf{a}_y + 0.5\mathbf{a}_z)/2.96}$$

1.3. The vector from the origin to the point A is given as $(6, -2, -4)$, and the unit vector directed from the origin toward point B is $(2, -2, 1)/3$. If points A and B are ten units apart, find the coordinates of point B.

With $\mathbf{A} = (6, -2, -4)$ and $\mathbf{B} = \frac{1}{3}B(2, -2, 1)$, we use the fact that $|\mathbf{B} - \mathbf{A}| = 10$, or

$|(6 - \frac{2}{3}B)\mathbf{a}_x - (2 - \frac{2}{3}B)\mathbf{a}_y - (4 + \frac{1}{3}B)\mathbf{a}_z| = 10$

Expanding, obtain

$36 - 8B + \frac{4}{9}B^2 + 4 - \frac{8}{3}B + \frac{4}{9}B^2 + 16 + \frac{8}{3}B + \frac{1}{9}B^2 = 100$

or $B^2 - 8B - 44 = 0$. Thus $B = \frac{8 \pm \sqrt{64-176}}{2} = 11.75$ (taking positive option) and so

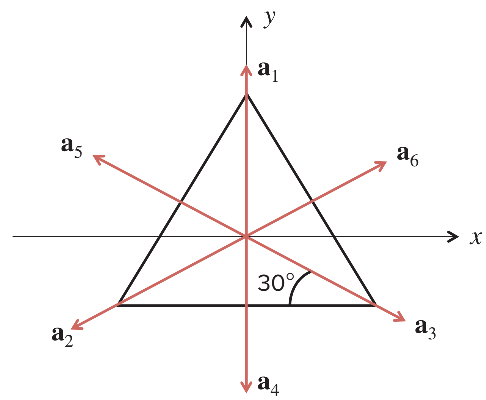
$$\mathbf{B} = \frac{2}{3}(11.75)\mathbf{a}_x - \frac{2}{3}(11.75)\mathbf{a}_y + \frac{1}{3}(11.75)\mathbf{a}_z = \underline{7.83\mathbf{a}_x - 7.83\mathbf{a}_y + 3.92\mathbf{a}_z}$$

1.4. A circle, centered at the origin with a radius of 2 units, lies in the xy plane. Determine the unit vector in rectangular components that lies in the xy plane, is tangent to the circle at $(\sqrt{3}, -1, 0)$, and is in the general direction of increasing values of x:

A unit vector tangent to this circle in the general increasing x direction is $\mathbf{t} = +\mathbf{a}_\phi$. Its x and y components are $\mathbf{t}_x = \mathbf{a}_\phi \cdot \mathbf{a}_x = -\sin\phi$, and $\mathbf{t}_y = \mathbf{a}_\phi \cdot \mathbf{a}_y = \cos\phi$. At the point $(\sqrt{3}, -1)$, $\phi = 330°$, and so $\mathbf{t} = -\sin 330° \mathbf{a}_x + \cos 330° \mathbf{a}_y = \underline{0.5(\mathbf{a}_x + \sqrt{3}\mathbf{a}_y)}$.

1.5. An equilateral triangle lies in the xy plane with its centroid at the origin. One vertex lies on the positive y axis.

a) Find unit vectors that are directed from the origin to the three vertices: Referring to the figure,the easy one is $\underline{\mathbf{a}_1 = \mathbf{a}_y}$. Then, $\mathbf{a}_2$ will have negative x and y components, and can be constructed as $\mathbf{a}_2 = G(-\mathbf{a}_x - \tan 30° \, \mathbf{a}_y)$ where $G = (1 + \tan 30°)^{1/2} = 0.87$. So finally $\underline{\mathbf{a}_2 = -0.87(\mathbf{a}_x + 0.58\mathbf{a}_y)}$. Then, $\mathbf{a}_3$ is the same as $\mathbf{a}_2$, but with the x component reversed: $\underline{\mathbf{a}_3 = 0.87(\mathbf{a}_x - 0.58\mathbf{a}_y)}$.



b) Find unit vectors that are directed from the origin to the three sides, intersecting these at right angles: These will be $\mathbf{a}_4$, $\mathbf{a}_5$, and $\mathbf{a}_6$ in the figure, which are in turn just the part a results, oppositely directed:

$\mathbf{a}_4 = -\mathbf{a}_1 = -\mathbf{a}_y$, $\mathbf{a}_5 = -\mathbf{a}_3 = -0.87(\mathbf{a}_x - 0.58\mathbf{a}_y)$, and $\mathbf{a}_6 = -\mathbf{a}_2 = +0.87(\mathbf{a}_x + 0.58\mathbf{a}_y)$.

1.6. Find the acute angle between the two vectors $\mathbf{A} = 2\mathbf{a}_x + \mathbf{a}_y + 3\mathbf{a}_z$ and $\mathbf{B} = \mathbf{a}_x - 3\mathbf{a}_y + 2\mathbf{a}_z$ by using the definition of:

a) the dot product: First, $\mathbf{A} \cdot \mathbf{B} = 2 - 3 + 6 = 5 = AB\cos\theta$, where $A = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$, and where $B = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$. Therefore $\cos\theta = 5/14$, so that $\theta = \underline{69.1°}$.

b) the cross product: Begin with

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 2 & 1 & 3 \\ 1 & -3 & 2 \end{vmatrix} = 11\mathbf{a}_x - \mathbf{a}_y - 7\mathbf{a}_z$$

and then $|\mathbf{A} \times \mathbf{B}| = \sqrt{11^2 + 1^2 + 7^2} = \sqrt{171}$. So now, with $|\mathbf{A} \times \mathbf{B}| = AB\sin\theta = \sqrt{171}$, find $\theta = \sin^{-1}\left(\sqrt{171}/14\right) = \underline{69.1°}$

1.7. Given the field $\mathbf{F} = x\mathbf{a}_x + y\mathbf{a}_y$. If $\mathbf{F} \cdot \mathbf{G} = 2xy$ and $\mathbf{F} \times \mathbf{G} = (x^2 - y^2)\,\mathbf{a}_z$, find $\mathbf{G}$:
Let $\mathbf{G} = g_1\,\mathbf{a}_x + g_2\,\mathbf{a}_y + g_3\,\mathbf{a}_z$ Then $\mathbf{F} \cdot \mathbf{G} = g_1 x + g_2 y = 2xy$, and

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ x & y & 0 \\ g_1 & g_2 & g_3 \end{vmatrix} = g_3 y\,\mathbf{a}_x - g_3 x\,\mathbf{a}_y + (g_2 x - g_1 y)\,\mathbf{a}_z = (x^2 - y^2)\,\mathbf{a}_z$$

From the last equation, it is clear that $g_3 = 0$, and that $g_1 = y$ and $g_2 = x$. This is confirmed in the $\mathbf{F} \cdot \mathbf{G}$ equation. So finally $\underline{\mathbf{G} = y\mathbf{a}_x + x\mathbf{a}_y}$.

1.8. Demonstrate the ambiguity that results when the cross product is used to find the angle between two vectors by finding the angle between $\mathbf{A} = 3\mathbf{a}_x - 2\mathbf{a}_y + 4\mathbf{a}_z$ and $\mathbf{B} = 2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z$. Does this ambiguity exist when the dot product is used?

We use the relation $\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}| \sin\theta \mathbf{n}$. With the given vectors we find

$$\mathbf{A} \times \mathbf{B} = 14\mathbf{a}_y + 7\mathbf{a}_z = 7\sqrt{5} \underbrace{\left[\frac{2\mathbf{a}_y + \mathbf{a}_z}{\sqrt{5}} \right]}_{\pm\mathbf{n}} = \sqrt{9+4+16}\sqrt{4+1+4} \, \sin\theta \, \mathbf{n}$$

where $\mathbf{n}$ is identified as shown; we see that $\mathbf{n}$ can be positive or negative, as $\sin\theta$ can be positive or negative. This apparent sign ambiguity is not the real problem, however, as we really want the magnitude of the angle anyway. Choosing the positive sign, we are left with $\sin\theta = 7\sqrt{5}/(\sqrt{29}\sqrt{9}) = 0.969$. *Two* values of θ (75.7° and 104.3°) satisfy this equation, and hence the real ambiguity.

In using the dot product, we find $\mathbf{A} \cdot \mathbf{B} = 6 - 2 - 8 = -4 = |\mathbf{A}||\mathbf{B}| \cos\theta = 3\sqrt{29} \cos\theta$, or $\cos\theta = -4/(3\sqrt{29}) = -0.248 \Rightarrow \theta = -75.7°$. Again, the minus sign is not important, as we care only about the angle magnitude. The main point is that *only one* θ value results when using the dot product, so no ambiguity.

1.9. A field is given as

$$\mathbf{G} = \frac{25}{(x^2 + y^2)}(x\mathbf{a}_x + y\mathbf{a}_y)$$

Find:

a) a unit vector in the direction of $\mathbf{G}$ at $P(3, 4, -2)$: Have $\mathbf{G}_p = 25/(9 + 16) \times (3, 4, 0) = 3\mathbf{a}_x + 4\mathbf{a}_y$, and $|\mathbf{G}_p| = 5$. Thus $\mathbf{a}_G = \underline{(0.6, 0.8, 0)}$.

b) the angle between $\mathbf{G}$ and $\mathbf{a}_x$ at P: The angle is found through $\mathbf{a}_G \cdot \mathbf{a}_x = \cos\theta$. So $\cos\theta = (0.6, 0.8, 0) \cdot (1, 0, 0) = 0.6$. Thus $\underline{\theta = 53°}$.

c) the value of the following double integral on the plane $y = 7$:

$$\int_0^4 \int_0^2 \mathbf{G} \cdot \mathbf{a}_y dz dx$$

$$\int_0^4 \int_0^2 \frac{25}{x^2 + y^2}(x\mathbf{a}_x + y\mathbf{a}_y) \cdot \mathbf{a}_y dz dx = \int_0^4 \int_0^2 \frac{25}{x^2 + 49} \times 7 \, dz dx = \int_0^4 \frac{350}{x^2 + 49} dx$$

$$= 350 \times \frac{1}{7} \left[\tan^{-1}\left(\frac{4}{7}\right) - 0 \right] = \underline{26}$$

1.10. By expressing diagonals as vectors and using the definition of the dot product, find the smaller angle between any two diagonals of a cube, where each diagonal connects diametrically opposite corners, and passes through the center of the cube:

Assuming a side length, b, two diagonal vectors would be $\mathbf{A} = b(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)$ and $\mathbf{B} = b(\mathbf{a}_x - \mathbf{a}_y + \mathbf{a}_z)$. Now use $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta$, or $b^2(1 - 1 + 1) = (\sqrt{3}b)(\sqrt{3}b) \cos\theta \Rightarrow \cos\theta = 1/3 \Rightarrow \theta = \underline{70.53°}$. This result (in magnitude) is the same for *any* two diagonal vectors.

1.11. Given the points $M(0.1, -0.2, -0.1)$, $N(-0.2, 0.1, 0.3)$, and $P(0.4, 0, 0.1)$, find:

a) the vector $\mathbf{R}_{MN}$: $\mathbf{R}_{MN} = (-0.2, 0.1, 0.3) - (0.1, -0.2, -0.1) = \underline{(-0.3, 0.3, 0.4)}$.

b) the dot product $\mathbf{R}_{MN} \cdot \mathbf{R}_{MP}$: $\mathbf{R}_{MP} = (0.4, 0, 0.1) - (0.1, -0.2, -0.1) = (0.3, 0.2, 0.2)$. $\mathbf{R}_{MN} \cdot \mathbf{R}_{MP} = (-0.3, 0.3, 0.4) \cdot (0.3, 0.2, 0.2) = -0.09 + 0.06 + 0.08 = \underline{0.05}$.

c) the scalar projection of $\mathbf{R}_{MN}$ on $\mathbf{R}_{MP}$:

$$\mathbf{R}_{MN} \cdot \mathbf{a}_{RMP} = (-0.3, 0.3, 0.4) \cdot \frac{(0.3, 0.2, 0.2)}{\sqrt{0.09 + 0.04 + 0.04}} = \frac{0.05}{\sqrt{0.17}} = \underline{0.12}$$

d) the angle between $\mathbf{R}_{MN}$ and $\mathbf{R}_{MP}$:

$$\theta_M = \cos^{-1}\left(\frac{\mathbf{R}_{MN} \cdot \mathbf{R}_{MP}}{|\mathbf{R}_{MN}||\mathbf{R}_{MP}|}\right) = \cos^{-1}\left(\frac{0.05}{\sqrt{0.34}\sqrt{0.17}}\right) = \underline{78^\circ}$$

1.12. Write an expression in rectangular components for the vector that extends from (x_1, y_1, z_1) to (x_2, y_2, z_2) and determine the magnitude of this vector.

The two points can be written as vectors from the origin:

$$\mathbf{A}_1 = x_1\mathbf{a}_x + y_1\mathbf{a}_y + z_1\mathbf{a}_z \quad \text{and} \quad \mathbf{A}_2 = x_2\mathbf{a}_x + y_2\mathbf{a}_y + z_2\mathbf{a}_z$$

The desired vector will now be the difference:

$$\mathbf{A}_{12} = \mathbf{A}_2 - \mathbf{A}_1 = (x_2 - x_1)\mathbf{a}_x + (y_2 - y_1)\mathbf{a}_y + (z_2 - z_1)\mathbf{a}_z$$

whose magnitude is

$$|\mathbf{A}_{12}| = \sqrt{\mathbf{A}_{12} \cdot \mathbf{A}_{12}} = \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{1/2}$$

1.13. a) Find the vector component of $\mathbf{F} = (10, -6, 5)$ that is parallel to $\mathbf{G} = (0.1, 0.2, 0.3)$:

$$\mathbf{F}_{||G} = \frac{\mathbf{F} \cdot \mathbf{G}}{|\mathbf{G}|^2}\mathbf{G} = \frac{(10, -6, 5) \cdot (0.1, 0.2, 0.3)}{0.01 + 0.04 + 0.09}(0.1, 0.2, 0.3) = \underline{(0.93, 1.86, 2.79)}$$

b) Find the vector component of $\mathbf{F}$ that is perpendicular to $\mathbf{G}$:

$$\mathbf{F}_{pG} = \mathbf{F} - \mathbf{F}_{||G} = (10, -6, 5) - (0.93, 1.86, 2.79) = \underline{(9.07, -7.86, 2.21)}$$

c) Find the vector component of $\mathbf{G}$ that is perpendicular to $\mathbf{F}$:

$$\mathbf{G}_{pF} = \mathbf{G} - \mathbf{G}_{||F} = \mathbf{G} - \frac{\mathbf{G} \cdot \mathbf{F}}{|\mathbf{F}|^2}\mathbf{F} = (0.1, 0.2, 0.3) - \frac{1.3}{100 + 36 + 25}(10, -6, 5) = \underline{(0.02, 0.25, 0.26)}$$

1.14. Given that $\mathbf{A}+\mathbf{B}+\mathbf{C}=0$, where the three vectors represent line segments and extend from a common origin,

a) must the three vectors be coplanar?

In terms of the components, the vector sum will be

$$\mathbf{A}+\mathbf{B}+\mathbf{C}=(A_x+B_x+C_x)\mathbf{a}_x+(A_y+B_y+C_y)\mathbf{a}_y+(A_z+B_z+C_z)\mathbf{a}_z$$

which we require to be zero. Suppose the coordinate system is configured so that vectors **A** and **B** lie in the x-y plane; in this case $A_z = B_z = 0$. Then C_z has to be zero in order for the three vectors to sum to zero. Therefore, the three vectors <u>must be coplanar</u>.

b) If $\mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{D}=0$, are the four vectors coplanar?

The vector sum is now

$$\mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{D}=(A_x+B_x+C_x+D_x)\mathbf{a}_x+(A_y+B_y+C_y+D_y)\mathbf{a}_y+(A_z+B_z+C_z+D_z)\mathbf{a}_z$$

Now, for example, if **A** and **B** lie in the x-y plane, **C** and **D** need not, as long as $C_z+D_z=0$. So the four vectors <u>need not be coplanar</u> to have a zero sum.

1.15. Three vectors extending from the origin are given as $\mathbf{r}_1 = (7,3,-2)$, $\mathbf{r}_2 = (-2,7,-3)$, and $\mathbf{r}_3 = (0,2,3)$. Find:

a) a unit vector perpendicular to both $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$\mathbf{a}_{p12} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{|\mathbf{r}_1 \times \mathbf{r}_2|} = \frac{(5,25,55)}{60.6} = \underline{(0.08, 0.41, 0.91)}$$

b) a unit vector perpendicular to the vectors $\mathbf{r}_1-\mathbf{r}_2$ and $\mathbf{r}_2-\mathbf{r}_3$: $\mathbf{r}_1-\mathbf{r}_2=(9,-4,1)$ and $\mathbf{r}_2-\mathbf{r}_3=(-2,5,-6)$. So $\mathbf{r}_1-\mathbf{r}_2\times\mathbf{r}_2-\mathbf{r}_3=(19,52,37)$. Then

$$\mathbf{a}_p = \frac{(19,52,37)}{|(19,52,37)|} = \frac{(19,52,37)}{66.6} = \underline{(0.29,0.78,0.56)}$$

c) the area of the triangle defined by $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$\text{Area} = \frac{1}{2}|\mathbf{r}_1 \times \mathbf{r}_2| = \underline{30.3}$$

d) the area of the triangle defined by the heads of $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$:

$$\text{Area} = \frac{1}{2}|(\mathbf{r}_2-\mathbf{r}_1)\times(\mathbf{r}_2-\mathbf{r}_3)| = \frac{1}{2}|(-9,4,-1)\times(-2,5,-6)| = \underline{33.3}$$

1.16. In geometrical optics, the path of a light ray is treated as a vector having the usual three components in a rectangular coordinate system. When light reflects from a plane surface, the effect is to reverse the vector component of the ray that is normal to the surface. This yields a reflection angle that is equal to the incidence angle. Explain what happens when light ray reflects from a *corner cube reflector*, consisting of three mutually orthogonal surfaces that occupy, for example, the xy, xz, and yz planes. The ray is incident in an arbitrary direction within the first octant of the coordinate system, and has negative x, y, and z directions of travel: We can model the incident ray as the vector, $\mathbf{R}_i = a(-\mathbf{a}_x) + b(-\mathbf{a}_y) + c(-\mathbf{a}_z)$, where a, b, and c are arbitrary positive values. With the three orthogonal planes positioned as given, their effect is to reverse all three components of the incident vector, giving $\mathbf{R}_r = a(\mathbf{a}_x) + b(\mathbf{a}_y) + c(\mathbf{a}_z) = -\mathbf{R}_i$. So the light propagates in precisely the reverse direction after reflection, no matter what direction it arrived from.

1.17. Point $A(-4, 2, 5)$ and the two vectors, $\mathbf{R}_{AM} = (20, 18, -10)$ and $\mathbf{R}_{AN} = (-10, 8, 15)$, define a triangle.

a) Find a unit vector perpendicular to the triangle: Use

$$\mathbf{a}_p = \frac{\mathbf{R}_{AM} \times \mathbf{R}_{AN}}{|\mathbf{R}_{AM} \times \mathbf{R}_{AN}|} = \frac{(350, -200, 340)}{527.35} = \underline{(0.664, -0.379, 0.645)}$$

The vector in the opposite direction to this one is also a valid answer.

b) Find a unit vector in the plane of the triangle and perpendicular to $\mathbf{R}_{AN}$:

$$\mathbf{a}_{AN} = \frac{(-10, 8, 15)}{\sqrt{389}} = (-0.507, 0.406, 0.761)$$

Then

$$\mathbf{a}_{pAN} = \mathbf{a}_p \times \mathbf{a}_{AN} = (0.664, -0.379, 0.645) \times (-0.507, 0.406, 0.761) = \underline{(-0.550, -0.832, 0.077)}$$

The vector in the opposite direction to this one is also a valid answer.

c) Find a unit vector in the plane of the triangle that bisects the interior angle at A: A non-unit vector in the required direction is $(1/2)(\mathbf{a}_{AM} + \mathbf{a}_{AN})$, where

$$\mathbf{a}_{AM} = \frac{(20, 18, -10)}{|(20, 18, -10)|} = (0.697, 0.627, -0.348)$$

Now

$$\frac{1}{2}(\mathbf{a}_{AM} + \mathbf{a}_{AN}) = \frac{1}{2}[(0.697, 0.627, -0.348) + (-0.507, 0.406, 0.761)] = (0.095, 0.516, 0.207)$$

Finally,

$$\mathbf{a}_{bis} = \frac{(0.095, 0.516, 0.207)}{|(0.095, 0.516, 0.207)|} = \underline{(0.168, 0.915, 0.367)}$$

1.18. Given two vector fields, $\mathbf{E} = (A/r)\sin\theta\,\mathbf{a}_\theta$ and $\mathbf{H} = (B/r)\sin\theta\,\mathbf{a}_\phi$, where A and B are constants,

a) evaluate $\mathbf{S} = \mathbf{E}\times\mathbf{H}$ and express the result in rectangular coordinates: Begin with

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} = \frac{AB}{r^2}\sin^2\theta\,\mathbf{a}_r$$

Now, find the three rectangular components by using:

$$S_x = \mathbf{S}\cdot\mathbf{a}_x = \frac{AB}{r^2}\sin^2\theta\,\mathbf{a}_r\cdot\mathbf{a}_x = AB\frac{x(x^2+y^2)}{(x^2+y^2+z^2)^{5/2}}$$

where we have used $\mathbf{a}_r\cdot\mathbf{a}_x = \sin\theta\cos\phi$ with $r = (x^2+y^2+z^2)^{1/2}$,
$\sin\theta = (x^2+y^2)^{1/2}/(x^2+y^2+z^2)^{1/2}$ and $\cos\phi = x/(x^2+y^2)^{1/2}$.
Then

$$S_y = \mathbf{S}\cdot\mathbf{a}_y = \frac{AB}{r^2}\sin^2\theta\,\mathbf{a}_r\cdot\mathbf{a}_y = AB\frac{y(x^2+y^2)}{(x^2+y^2+z^2)^{5/2}}$$

where we have used $\mathbf{a}_r\cdot\mathbf{a}_y = \sin\theta\sin\phi$ with $\sin\phi = y/(x^2+y^2)^{1/2}$.
Finally

$$S_z = \mathbf{S}\cdot\mathbf{a}_z = \frac{AB}{r^2}\sin^2\theta\,\mathbf{a}_r\cdot\mathbf{a}_z = AB\frac{z(x^2+y^2)}{(x^2+y^2+z^2)^{5/2}}$$

where $\mathbf{a}_r\cdot\mathbf{a}_z = \cos\theta = z/(x^2+y^2+z^2)^{1/2}$.

b) Determine $\mathbf{S}$ along the x, y, and z axes: Using the part a results, we would have

$$\mathbf{S}(x,0,0) = \frac{AB}{x^2}\,\mathbf{a}_x, \quad \mathbf{S}(0,y,0) = \frac{AB}{y^2}\,\mathbf{a}_y, \quad \mathbf{S}(0,0,z) = 0$$

1.19. Consider the important inverse-square dependent radial field in spherical coordinates: $\mathbf{F} = A/r^2\,\mathbf{a}_r$ where A is a constant.

a) Transform the given field into cylindrical coordinates: Using $r^2 = \rho^2 + z^2$, the given field may be written:

$$\mathbf{F} = \frac{A}{\rho^2+z^2}\left[f_\rho\,\mathbf{a}_\rho + f_\phi\,\mathbf{a}_\phi + f_z\,\mathbf{a}_z\right]$$

Now

$$f_\rho = \mathbf{a}_r\cdot\mathbf{a}_\rho = \sin\theta = \frac{\rho}{r} = \frac{\rho}{(\rho^2+z^2)^{1/2}}$$

$$f_\phi = \mathbf{a}_r\cdot\mathbf{a}_\phi = 0$$

$$f_z = \mathbf{a}_r\cdot\mathbf{a}_z = \cos\theta = \frac{z}{r} = \frac{z}{(\rho^2+z^2)^{1/2}}$$

Substituting all, we find

$$\underline{\mathbf{F} = \frac{A}{\rho^2+z^2}\left[\rho\mathbf{a}_\rho + z\mathbf{a}_z\right]}$$

b) Transform the given field into rectangular coordinates: The quickest way is to use the results of part a by using $\rho = \sqrt{x^2+y^2}$ and $\mathbf{a}_\rho = \cos\phi\,\mathbf{a}_x + \sin\phi\,\mathbf{a}_y$, where $\cos\phi = x/\sqrt{x^2+y^2}$ and $\sin\phi = y/\sqrt{x^2+y^2}$. Incorporating the above leads to:

$$\underline{\mathbf{F} = \frac{A}{(x^2+y^2+z^2)^{3/2}}\left[x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z\right]}$$

1.20. If the three sides of a triangle are represented by the vectors **A**, **B**, and **C**, all directed counter-clockwise, show that $|\mathbf{C}|^2 = (\mathbf{A}+\mathbf{B}) \cdot (\mathbf{A}+\mathbf{B})$ and expand the product to obtain the law of cosines.

With the vectors drawn as described above, we find that $\mathbf{C} = -(\mathbf{A}+\mathbf{B})$ and so $|\mathbf{C}|^2 = C^2 = \mathbf{C} \cdot \mathbf{C} = (\mathbf{A}+\mathbf{B}) \cdot (\mathbf{A}+\mathbf{B})$ So far so good. Now if we expand the product, obtain

$$(\mathbf{A}+\mathbf{B}) \cdot (\mathbf{A}+\mathbf{B}) = A^2 + B^2 + 2\mathbf{A} \cdot \mathbf{B}$$

where $\mathbf{A} \cdot \mathbf{B} = AB\cos(180° - \alpha) = -AB\cos\alpha$ where α is the interior angle at the junction of **A** and **B**. Using this, we have $C^2 = A^2 + B^2 - 2AB\cos\alpha$, which is the law of cosines.

1.21. Express in cylindrical components:

a) the vector from $C(3, 2, -7)$ to $D(-1, -4, 2)$:
$C(3, 2, -7) \to C(\rho = 3.61, \phi = 33.7°, z = -7)$ and
$D(-1, -4, 2) \to D(\rho = 4.12, \phi = -104.0°, z = 2)$.
Now $\mathbf{R}_{CD} = (-4, -6, 9)$ and $R_\rho = \mathbf{R}_{CD} \cdot \mathbf{a}_\rho = -4\cos(33.7) - 6\sin(33.7) = -6.66$. Then $R_\phi = \mathbf{R}_{CD} \cdot \mathbf{a}_\phi = 4\sin(33.7) - 6\cos(33.7) = -2.77$. So $\mathbf{R}_{CD} = \underline{-6.66\mathbf{a}_\rho - 2.77\mathbf{a}_\phi + 9\mathbf{a}_z}$

b) a unit vector at D directed toward C:
$\mathbf{R}_{CD} = (4, 6, -9)$ and $R_\rho = \mathbf{R}_{DC} \cdot \mathbf{a}_\rho = 4\cos(-104.0) + 6\sin(-104.0) = -6.79$. Then $R_\phi = \mathbf{R}_{DC} \cdot \mathbf{a}_\phi = 4[-\sin(-104.0)] + 6\cos(-104.0) = 2.43$. So $\mathbf{R}_{DC} = -6.79\mathbf{a}_\rho + 2.43\mathbf{a}_\phi - 9\mathbf{a}_z$
Thus $\mathbf{a}_{DC} = \underline{-0.59\mathbf{a}_\rho + 0.21\mathbf{a}_\phi - 0.78\mathbf{a}_z}$

c) a unit vector at D directed toward the origin: Start with $\mathbf{r}_D = (-1, -4, 2)$, and so the vector toward the origin will be $-\mathbf{r}_D = (1, 4, -2)$. Thus in cartesian the unit vector is $\mathbf{a} = (0.22, 0.87, -0.44)$. Convert to cylindrical:
$a_\rho = (0.22, 0.87, -0.44) \cdot \mathbf{a}_\rho = 0.22\cos(-104.0) + 0.87\sin(-104.0) = -0.90$, and
$a_\phi = (0.22, 0.87, -0.44) \cdot \mathbf{a}_\phi = 0.22[-\sin(-104.0)] + 0.87\cos(-104.0) = 0$, so that finally, $\mathbf{a} = \underline{-0.90\mathbf{a}_\rho - 0.44\mathbf{a}_z}$.

1.22. A sphere of radius a, centered at the origin, rotates about the z axis at angular velocity Ω rad/s. The rotation direction is clockwise when one is looking in the positive z direction.

a) Using spherical components, write an expression for the velocity field, **v**, which gives the tangential velocity at any point on the sphere surface:

The tangential velocity is the product of the angular velocity and the perpendicular distance from the rotation axis. With clockwise rotation, and with sphere radius a, we obtain

$$\mathbf{v}(\theta) = \underline{\Omega a \sin\theta \, \mathbf{a}_\phi \text{ m/s}}$$

b) Derive an expression for the difference in velocities between two points on the surface having different latitudes, where the latitude difference is $\Delta\theta$ in radians. Assume $\Delta\theta$ is small:

Method 1: The velocity difference between the two points can be approximated as

$$\Delta v \doteq \Delta\theta \left.\frac{dv}{d\theta}\right|_{\theta_0}$$

where θ_0 is the mean angle between the two points, and where $\Delta\theta$ is small enough so that $v(\theta)$ can be approximated as a linear function. Using this along with the part a result, we find:

$$\Delta v \doteq \underline{a\Omega\Delta\theta\cos\theta_0 \text{ m/s}}$$

1.22 b) (continued) Method 2 (harder): Write

$$\Delta v = a\Omega(\sin\theta_2 - \sin\theta_1)$$

where θ_1 and θ_2 lie on either side of θ_0, and where $\theta_2 - \theta_1 = \Delta\theta$. Using a trig identity, the above expression becomes

$$\Delta v = 2a\Omega \underbrace{\cos\left(\frac{1}{2}(\theta_2+\theta_1)\right)}_{\cos\theta_0} \underbrace{\sin\left(\frac{1}{2}(\theta_2-\theta_1)\right)}_{\doteq \Delta\theta/2} \doteq \underline{a\Omega\Delta\theta\cos\theta_0 \text{ m/s}}$$

c) Find the difference in velocities at locations $\pm 1.0°$ on either side of 45° north latitude on Earth. Take the Earth's radius as 6370 km at 45°:

We have $a = 6370$ km, $\theta_0 = 45°$ (same in spherical and geographic coordinates), $\Delta\theta = 2.0° = 0.035$ rad, and $\Omega = 2\pi/24\,\text{hr} = 7.3 \times 10^{-5}$ rad/s. Using part *b*, the velocity difference becomes

$$\Delta v \doteq (6.37 \times 10^6\,\text{m})(7.3 \times 10^{-5}\,\text{rad/s})(0.035\,\text{rad})(\sqrt{2}/2) = \underline{11.5\,\text{m/s}}$$

with speed increasing southward.

1.23. The surfaces $\rho = 3$, $\rho = 5$, $\phi = 100°$, $\phi = 130°$, $z = 3$, and $z = 4.5$ define a closed surface.

a) Find the enclosed volume:

$$\text{Vol} = \int_3^{4.5}\int_{100°}^{130°}\int_3^5 \rho\, d\rho\, d\phi\, dz = \underline{6.28}$$

NOTE: The limits on the ϕ integration must be converted to radians (as was done here, but not shown).

b) Find the total area of the enclosing surface:

$$\text{Area} = 2\int_{100°}^{130°}\int_3^5 \rho\, d\rho\, d\phi + \int_3^{4.5}\int_{100°}^{130°} 3\, d\phi\, dz + \int_3^{4.5}\int_{100°}^{130°} 5\, d\phi\, dz + 2\int_3^{4.5}\int_3^5 d\rho\, dz = \underline{20.7}$$

c) Find the total length of the twelve edges of the surfaces:

$$\text{Length} = 4 \times 1.5 + 4 \times 2 + 2 \times \left[\frac{30°}{360°} \times 2\pi \times 3 + \frac{30°}{360°} \times 2\pi \times 5\right] = \underline{22.4}$$

d) Find the length of the longest straight line that lies entirely within the volume: This will be between the points A($\rho = 3$, $\phi = 100°$, $z = 3$) and B($\rho = 5$, $\phi = 130°$, $z = 4.5$). Performing point transformations to cartesian coordinates, these become A($x = -0.52$, $y = 2.95$, $z = 3$) and B($x = -3.21$, $y = 3.83$, $z = 4.5$). Taking A and B as vectors directed from the origin, the requested length is

$$\text{Length} = |\mathbf{B} - \mathbf{A}| = |(-2.69, 0.88, 1.5)| = \underline{3.21}$$

1.24. Two unit vectors, $\mathbf{a}_1$ and $\mathbf{a}_2$ lie in the xy plane and pass through the origin. They make angles ϕ_1 and ϕ_2 with the x axis respectively.

a) Express each vector in rectangular components; Have $\mathbf{a}_1 = A_{x1}\mathbf{a}_x + A_{y1}\mathbf{a}_y$, so that $A_{x1} = \mathbf{a}_1 \cdot \mathbf{a}_x = \cos\phi_1$. Then, $A_{y1} = \mathbf{a}_1 \cdot \mathbf{a}_y = \cos(90 - \phi_1) = \sin\phi_1$. Therefore,

$$\mathbf{a}_1 = \cos\phi_1\,\mathbf{a}_x + \sin\phi_1\,\mathbf{a}_y \quad \text{and similarly,} \quad \mathbf{a}_2 = \cos\phi_2\,\mathbf{a}_x + \sin\phi_2\,\mathbf{a}_y$$

b) take the dot product and verify the trigonometric identity, $\cos(\phi_1 - \phi_2) = \cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2$: From the definition of the dot product,

$$\begin{aligned}\mathbf{a}_1 \cdot \mathbf{a}_2 &= (1)(1)\cos(\phi_1 - \phi_2)\\ &= (\cos\phi_1\,\mathbf{a}_x + \sin\phi_1\,\mathbf{a}_y)\cdot(\cos\phi_2\,\mathbf{a}_x + \sin\phi_2\,\mathbf{a}_y) = \cos\phi_1\,\cos\phi_2 + \sin\phi_1\,\sin\phi_2\end{aligned}$$

c) take the cross product and verify the trigonometric identity $\sin(\phi_2 - \phi_1) = \sin\phi_2\cos\phi_1 - \cos\phi_2\sin\phi_1$: From the definition of the cross product, and since $\mathbf{a}_1$ and $\mathbf{a}_2$ both lie in the x-y plane,

$$\mathbf{a}_1 \times \mathbf{a}_2 = (1)(1)\sin(\phi_1 - \phi_2)\,\mathbf{a}_z = \begin{vmatrix}\mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z\\ \cos\phi_1 & \sin\phi_1 & 0\\ \cos\phi_2 & \sin\phi_2 & 0\end{vmatrix}$$
$$= \left[\sin\phi_2\cos\phi_1 - \cos\phi_2\sin\phi_1\right]\,\mathbf{a}_z$$

thus verified.

1.25. Convert the vector field $\mathbf{H} = A(x^2 + y^2)^{-1}[x\,\mathbf{a}_y - y\,\mathbf{a}_x]$ into cylindrical coordinates. A is a constant: The z component is obviously zero, so we are left only with the ρ and ϕ components to find. First,

$$H_\rho = \mathbf{H}\cdot\mathbf{a}_\rho = \frac{A}{x^2+y^2}\left[x\,\mathbf{a}_y\cdot\mathbf{a}_\rho - y\,\mathbf{a}_x\cdot\mathbf{a}_\rho\right]$$

where $\mathbf{a}_y \cdot \mathbf{a}_\rho = \sin\phi = y/\sqrt{x^2+y^2}$ and $\mathbf{a}_x \cdot \mathbf{a}_\rho = \cos\phi = x/\sqrt{x^2+y^2}$. Thus

$$H_\rho = \frac{A}{(x^2+y^2)^{3/2}}\,[xy - yx] = \underline{0}$$

Then

$$H_\phi = \mathbf{H}\cdot\mathbf{a}_\phi = \frac{A}{x^2+y^2}\left[x\,\mathbf{a}_y\cdot\mathbf{a}_\phi - y\,\mathbf{a}_x\cdot\mathbf{a}_\phi\right]$$

where $\mathbf{a}_y \cdot \mathbf{a}_\phi = \cos\phi = x/\sqrt{x^2+y^2}$ and $\mathbf{a}_x \cdot \mathbf{a}_\phi = -\sin\phi = -y/\sqrt{x^2+y^2}$. Thus

$$H_\phi = \frac{A}{(x^2+y^2)^{3/2}}\left[x^2+y^2\right] = \frac{A}{(x^2+y^2)^{1/2}} = \frac{A}{\rho}$$

Finally

$$\underline{\mathbf{H} = \frac{A}{\rho}\,\mathbf{a}_\phi}$$

1.26. Express the uniform vector field, $\mathbf{F} = 10\,\mathbf{a}_y$ in

a) cylindrical components:
$F_\rho = 10\,\mathbf{a}_y \cdot \mathbf{a}_\rho = 10 \sin\phi$, $F_\phi = 10\,\mathbf{a}_y \cdot \mathbf{a}_\phi = 10\cos\phi$, and $F_z = 0$.
Combining, we obtain $\underline{\mathbf{F}(\rho,\phi) = 10(\sin\phi\,\mathbf{a}_\rho + \cos\phi\,\mathbf{a}_\phi)}$.

b) spherical components:
$F_r = 10\,\mathbf{a}_y \cdot \mathbf{a}_r = 10\sin\theta\sin\phi$; $F_\theta = 10\,\mathbf{a}_y \cdot \mathbf{a}_\theta = 10\cos\theta\sin\phi$; $F_\phi = 10\,\mathbf{a}_y \cdot \mathbf{a}_\phi = 10\cos\phi$.
Combining, we obtain $\underline{\mathbf{F}(r,\theta,\phi) = 10\left[\sin\theta\sin\phi\,\mathbf{a}_r + \cos\theta\sin\phi\,\mathbf{a}_\theta + \cos\phi\,\mathbf{a}_\phi\right]}$.

1.27. The dipole field is given in spherical coordinates as:

$$\mathbf{E} = \frac{A}{r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta)$$

where A is a constant and where $r > 0$.

a) Identify the surface on which the field is entirely perpendicular to the xy plane, and express the field on that surface in cylindrical coordinates: As there is no $\mathbf{a}_\phi$ component, the cylindrical coordinate field components will be $\mathbf{a}_\rho$ and $\mathbf{a}_z$ only. We look for the surface on which the $\mathbf{a}_\rho$ component is zero. So we set up:

$$E_\rho = \mathbf{E}\cdot\mathbf{a}_\rho = \frac{A}{r^3}(2\cos\theta\,\underbrace{\mathbf{a}_r\cdot\mathbf{a}_\rho}_{\sin\theta} + \sin\theta\,\underbrace{\mathbf{a}_\theta\cdot\mathbf{a}_\rho}_{\cos\theta}) = 0$$

From which the condition for zero E_ρ is identified:

$$3\cos\theta\sin\theta = 0 \;\Rightarrow\; \theta = 0, 90^\circ$$

The $\theta = 0$ option is only a line (the z axis – and the answer to part *b*), so the surface on which the field is perpendicular to the xy plane is the xy plane itself, on which $\theta = 90^\circ$. On that surface:

$$\mathbf{E}(\theta = 90) = \frac{A}{r^3}\,\mathbf{a}_\theta = -\frac{A}{\rho^3}\,\mathbf{a}_z$$

b) Identify the coordinate axis on which the field is entirely perpendicular to the xy plane and express the field there in cylindrical coordinates: This will be the $\theta = 0$ solution in part *a*, which is the z axis. The field on that axis is

$$\mathbf{E}(\theta = 0) = \frac{2A}{z^3}\,\mathbf{a}_z$$

c) Specify the surface on which the field is entirely parallel to the xy plane: In this case, we require a zero z component, so we construct:

$$E_z = \mathbf{E}\cdot\mathbf{a}_z = \frac{A}{r^3}(2\cos\theta\,\underbrace{\mathbf{a}_r\cdot\mathbf{a}_z}_{\cos\theta} + \sin\theta\,\underbrace{\mathbf{a}_\theta\cdot\mathbf{a}_z}_{-\sin\theta}) = 0$$

This tells us that for a zero z component, we must have

$$2\cos^2\theta - \sin^2\theta = 0 \;\Rightarrow\; 3\cos^2\theta - 1 = 0 \;\Rightarrow\; \theta = 54.74^\circ$$

The surface is a cone of angle 54.74° to the z axis, and extending over all values of r within the limits of validity of the given field.

1.28. State whether or not $\mathbf{A} = \mathbf{B}$ and, if not, what conditions are imposed on **A** and **B** when

a) $\mathbf{A} \cdot \mathbf{a}_x = \mathbf{B} \cdot \mathbf{a}_x$: For this to be true, both **A** and **B** must be oriented at the same angle, θ, from the x axis. But this would allow either vector to lie anywhere along a conical surface of angle θ about the x axis. Therefore, **A** *can* be equal to **B**, but not necessarily.

b) $\mathbf{A} \times \mathbf{a}_x = \mathbf{B} \times \mathbf{a}_x$: This is a more restrictive condition because the cross product gives a vector. For both cross products to lie in the same direction, **A**, **B**, and $\mathbf{a}_x$ must be coplanar. But if **A** lies at angle θ to the x axis, **B** could lie at θ *or* at $180° - \theta$ to give the same cross product. So again, **A** *can* be equal to **B**, but not necessarily.

c) $\mathbf{A} \cdot \mathbf{a}_x = \mathbf{B} \cdot \mathbf{a}_x$ *and* $\mathbf{A} \times \mathbf{a}_x = \mathbf{B} \times \mathbf{a}_x$: In this case, we need to satisfy both requirements in parts *a* and *b* – that is, **A**, **B**, and $\mathbf{a}_x$ must be coplanar, *and* **A** and **B** must lie at the same angle, θ, to $\mathbf{a}_x$. With coplanar vectors, this latter condition might imply that both $+\theta$ and $-\theta$ would therefore work. But the negative angle reverses the direction of the cross product direction. Therefore both vectors must lie in the same plane *and* lie at the same angle to x; i.e., **A** *must* be equal to **B**.

d) $\mathbf{A}\cdot\mathbf{C} = \mathbf{B}\cdot\mathbf{C}$ *and* $\mathbf{A}\times\mathbf{C} = \mathbf{B}\times\mathbf{C}$ where **C** is any vector except $\mathbf{C} = 0$: This is just the general case of part c. Since we can orient our coordinate system in any manner we choose, we can arrange it so that the x axis coincides with the direction of vector **C**. Thus all the arguments of part c apply, and again we conclude that **A** *must* be equal to **B**.

1.29. A vector field is expressed as $\mathbf{F} = 10z\dot{\mathbf{a}}_z$. Evaluate the components of this field that are *a*) normal, and *b*) tangent to a spherical surface of radius a:

a) The normal component is the radial component, found by projecting **F** into the $\mathbf{a}_r$ direction. In general,

$$F_r = \mathbf{F} \cdot \mathbf{a}_r = 10z\mathbf{a}_z \cdot \mathbf{a}_r = 10z \cos\theta$$

where, on the sphere surface, $z = a\cos\theta$. Therefore F_r (on surface) $= \underline{10a\cos^2\theta}$.

b) The tangential component is found by projecting **F** into the $\mathbf{a}_\theta$ or $\mathbf{a}_\phi$ directions. Note that $\mathbf{a}_z \cdot \mathbf{a}_\phi = 0$, so we are left with the tangential component lying in only the direction of $\mathbf{a}_\theta$:

$$F_\theta = 10z\mathbf{a}_z \cdot \mathbf{a}_\theta|_{r=a} = \underline{-10a\cos\theta\sin\theta}$$

1.30. Consider a problem analogous to the varying wind velocities encountered by transcontinental aircraft. We assume a constant altitude, a plane earth, a flight along the x axis from 0 to 10 units, no vertical velocity component, and no change in wind velocity with time. Assume $\mathbf{a}_x$ to be directed to the east and $\mathbf{a}_y$ to the north. The wind velocity at the operating altitude is assumed to be:

$$\mathbf{v}(x, y) = \frac{(0.01x^2 - 0.08x + 0.66)\mathbf{a}_x - (0.05x - 0.4)\mathbf{a}_y}{1 + 0.5y^2}$$

a) Determine the location and magnitude of the maximum tailwind encountered: Tailwind would be x-directed, and so we look at the x component only. Over the flight range, this function maximizes at a value of $0.86/(1 + 0.5y^2)$ at $x = 10$ (at the end of the trip). It reaches a local minimum of $0.50/(1 + 0.5y^2)$ at $x = 4$, and has another local maximum of $0.66/(1 + 0.5y^2)$ at the trip start, $x = 0$.

b) Repeat for headwind: The x component is always positive, and so therefore no headwind exists over the travel range.

c) Repeat for crosswind: Crosswind will be found from the y component, which is seen to maximize over the flight range at a value of $0.4/(1 + 0.5y^2)$ at the trip start ($x = 0$).

d) Would more favorable tailwinds be available at some other latitude? If so, where? Minimizing the denominator accomplishes this; in particular, the lattitude associated with $y = 0$ gives the strongest tailwind.